Algebraic analysis of V-cycle multigrid

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Abstract

We consider multigrid methods for symmetric positive definite linear systems. We develop an algebraic analysis of V-cycle schemes with Galerkin coarse grid matrices. This analysis is based on the Successive Subspace Correction convergence theory which we revisit. We reformulate it in a purely algebraic way, and extend its scope of application to, e.g., algebraic multigrid methods. This reformulation also yields more accurate bounds. Considering a model problem, we show that our results can give a satisfactorily sharp prediction of actual multigrid convergence.

Key words. multigrid, V-cycle, successive subspace correction, convergence analysis

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1 Introduction

We consider multigrid methods for solving symmetric positive definite (SPD) $n \times n$ linear systems

$$Ax = b. \tag{1.1}$$

Multigrid methods are based on the recursive use of a two-grid scheme. A basic two-grid method combines the action of a *smoother*, often a simple iterative method such as Gauss-Seidel, and a *coarse grid correction*, which involves solving a smaller problem on a coarser grid. A V-cycle multigrid method is obtained when this coarse problem is solved approximately with 1 iteration of the two-grid scheme on that level, and so on, until the coarsest level on which an exact solve is performed. Other cycles may be defined, for instance the W-cycle based on two stationary iterations at each level, see, e.g., [22].

Multigrid methods have been analyzed in several ways. Optimal convergence properties (with respect to the number of levels) can be proved via so-called smoothing and approximation properties, or via the theory of *Successive Subspace Correction* (SSC) methods (using the multilevel splitting of finite element spaces), see, e.g., [3, 5, 10, 11, 12, 13, 14, 18, 19, 26, 30]. However, bounds derived in this way do not, in general, give satisfactorily sharp predictions of actual multigrid convergence [22, p. 96]. Of course, the latter comment does not apply to the analysis in [27], which provides a sharp identity. However, to our knowledge, so far attempts to apply this identity to V-cycle multigrid methods have only lead to qualitative results.

On the other hand, accurate bounds may be obtained for two-grid methods, either by means of Fourier analysis [21, 22, 25], or using some appropriate algebraic tools [6, 8, 9, 16, 20]. In particular, the latter approach may be applied to algebraic multigrid (AMG) methods. This focus on two-grid schemes is motivated by the fact that, "if the two-grid method converges sufficiently well, then the multigrid method with W-cycle will have similar convergence properties" [22, p. 77] (see also [4, pp. 226–228] and [17]). However, V-cycle multigrid schemes are often preferred to W-cycle in practice, although, so far, no theoretical result seems sufficiently sharp to prove their actual potentialities. Moreover, there are known examples where the two-grid method converges relatively well, whereas the V-cycle multigrid scheme scales poorly with the number of levels [15]. Hence V-cycle analysis has to be, to some point, essentially different from two-grid analysis.

In this paper, we develop an algebraic analysis of V-cycle multigrid inspired from the SSC convergence theory as stated in [26, 30]. Whereas the original formulation seems only applicable to finite elements discretizations resulting from the refinement of a given coarse problem, our algebraic reformulation is applicable to any linear system (1.1) solved with a multigrid method satisfying the basic assumptions stated in the next section. Moreover, our approach also results in sharper bounds. Applied to the model Poisson problem with damped Jacobi smoothing, we show that the convergence factor is below 0.625 for the multigrid method with V-cycle, which is not far from the two-grid rate, approximately equal to 0.391. To our knowledge, this is the first time that one proves such a sharp estimate for V-cycle multigrid.

Alternatively, we could have taken inspiration from the theories developed in [10, 27], which enhance the SSC convergence theory in [26, 30]. However, the approach followed here seems to us the shortest path to bounds like those obtained this paper, with self-contained proofs based on matrix algebra.

The reminder of this paper is organized as follows. In Section 2, we state the general setting of this study and gather the needed assumptions. The theoretical analysis is developed in Section 3, and is illustrated on an example in Section 4.

Notation

Let I denote the identity matrix and O the zero matrix. When the dimensions are not obvious from the context, we write more specifically I_m for the $m \times m$ identity matrix, and $O_{m\times l}$ for the $m \times l$ zero matrix.

For any square matrix C, $\sigma(C)$ is its spectrum and $\rho(C)$ is its spectral radius (that is, its largest eigenvalue in modulus); $\|C\| = \sqrt{\rho(C^TC)}$ is the usual 2–norm, $\|C\|_F = \sqrt{\sum_{i,j} C_{ij}^2}$ the Frobenius norm, and $\|C\|_A = \|A^{1/2}CA^{-1/2}\|$ is the A–norm (often referred to as energy norm).

2 General setting

We consider a multigrid method with $J+1$ levels ($J \geq 1$); index J refers to the finest level (on which the system (1.1) is to be solved), and index 0 to the coarsest level. The number of unknowns at level k, $0 \leq k \leq J$, is denoted by n_k (thus $n_J = n$).

Our analysis applies to symmetric multigrid schemes based on the Galerkin principle for the SPD system (1.1); that is, the restriction is the transpose of the prolongation and the matrix A_k at level k, $k = J-1, \ldots, 0$, is given by $A_k = P_k^T A_{k+1} P_k$, where P_k is the prolongation from level k to level $k+1$; we also assume that the smoother R_k is SPD and that the number of pre-smoothing steps ν_k is equal to the number of post-smoothing steps. The algorithm for V-cycle multigrid is then as follows.

Multigrid with V-cycle at level k: $x_{n+1} = \mathrm{MG}(b, A_k, x_n, k)$

(1) Relax ν_k times with smoother R_k: $\bar{x}_n = Smooth(x_n, A_k, R_k, \nu_k, b)$

(2) Compute residual: $r_k = b - A_k \bar{x}_n$
(3) Restrict residual: $r_{k-1} = P_{k-1}^T r_k$
(4) Coarse grid correction: **if** $k = 1$, $e_0 = A_0^{-1} r_0$
else $e_{k-1} = \mathrm{MG}(r_{k-1}, A_{k-1}, 0, k-1)$
(5) Prolongate coarse grid correction: $\hat{\bar{x}}_n = \bar{x}_n + P_{k-1} e_{k-1}$
(6) Relax ν_k times with smoother $R_k :$ $x_{n+1} = Smooth(\hat{\bar{x}}_n, A_k, R_k, \nu_k, b)$

When applying this algorithm the error satisfies

$$A_k^{-1} b - x_{n+1} = E_{MG}^{(k)} \left(A_k^{-1} b - x_n \right)$$

where the iteration matrix $E_{MG}^{(k)}$ is recursively defined from

$$\begin{aligned} &E_{MG}^{(0)} = O \quad \text{and, for } \; k = 1, 2, \ldots, J \;\; : \\ &E_{MG}^{(k)} = (I - R_k^{-1} A_k)^{\nu_k} \left(I - P_{k-1} (I - E_{MG}^{(k-1)}) A_{k-1}^{-1} P_{k-1}^T A_k \right) (I - R_k^{-1} A_k)^{\nu_k} \end{aligned} \tag{2.1}$$

(see, e.g., [22, p. 48]). Our main objective is the analysis of the spectral radius of $E_{MG}^{(J)}$, which governs the convergence on the finest level. Our analysis makes uses of the following general assumptions.

General assumptions

- $n = n_J > n_{J-1} > \cdots > n_0$;
- P_k is an $n_{k+1} \times n_k$ matrix of rank n_k , $k = J-1, \ldots, 0$;
- $A_J = A$ and $A_k = P_k^T A_{k+1} P_k$, $k = J-1, \ldots, 0$;
- R_k is SPD and such that $\rho(I - R_k^{-1} A_k) < 1$, $k = J, \ldots, 1$.

Note also that most of our results do not refer explicitly to R_k , but are stated with respect to the matrices M_k defined from

$$I - M_k^{-1} A_k = (I - R_k^{-1} A_k)^{\nu_k} \; . \tag{2.2}$$

That is, M_k is the smoother which provides in 1 step the same effect as ν_k steps with R_k . The results stated with respect to M_k may then be seen as results stated for the case of 1 pre- and 1 post-smoothing steps, which can be extended to the general case via the relations (2.2).

We also define

$$\omega = \max \left(1 \, , \max_{1 \le k \le J} \; \max_{w_k \in \mathbf{R}^{n_k}} \; \frac{w_k^T A_k w_k}{w_k^T M_k^{(\nu)} w_k} \right) \; , \tag{2.3}$$

$$\omega_1 = \max \left(1 \, , \max_{1 \le k \le J} \; \max_{w_k \in \mathbf{R}^{n_k}} \; \frac{w_k^T A_k w_k}{w_k^T R_k w_k} \right) \; . \tag{2.4}$$

Our assumption $\rho\left(I - R_k^{-1} A_k\right) < 1$ implies that $\omega, \omega_1 < 2$.

Another useful definition is

$$\begin{aligned} \check{P}_J &= I \\ \check{P}_k &= \check{P}_{k+1}\, P_k \quad , \quad k = J-1, \dots, 0 \ , \end{aligned} \tag{2.5}$$

which can be viewed as prolongation matrices from level k to the finest level.

3 Theoretical results

Our main result is stated in Theorem 3.1 below. This result is inspired by Theorem 4.4 and Lemma 4.6 in [26], and Theorem 5.1 in [30]. Our, approach, however, is purely algebraic. The general lines of the proof follow the general lines of the original proofs in [26, 30], but the technical developments needed in between the major substeps are significantly different.

Theorem 3.1 contains some degrees of freedom, namely the matrices G_k , $k = J-1, \dots, 0$, which may be seen as pseudo restriction since G_k has the size of P_k^T . One may freely choose these matrices so as to minimize the bound on the spectral radius. Two concrete choices for G_k are discussed later in this section. Similarly to (2.5), we also define

$$\begin{aligned} \check{G}_J &= I \\ \check{G}_k &= G_k\, \check{G}_{k+1} \quad , \quad k = J-1, \dots, 0 \ . \end{aligned} \tag{3.1}$$

Theorem 3.1 *Let $E_{MG}^{(J)}$ be defined by (2.1) with P_k , $k = 0, \dots, J-1$, A_k , $k = 0, \dots, J$, and R_k , $k = 1, \dots, J$ satisfying the general assumptions stated in Section 2. For $k = 1, \dots, J$, let M_k be defined by* (2.2)*, and set $M_0 = A_0$.*

Let G_k , $k = 0, \dots, J-1$, be $n_k \times n_{k+1}$ matrices, and, for $k = 0, \dots, J$, let $\check{P}_k$ and $\check{G}_k$ be defined by, respectively, (2.5) *and* (3.1)*. Set $P_{-1} = G_{-1} = O_{n_0 \times n_0}$.*

Let

$$K = \max_{v \in \mathbf{R}^n} \frac{\sum_{k=0}^{J} v^T \check{G}_k^T (I - P_{k-1} G_{k-1})^T M_k (I - P_{k-1} G_{k-1}) \check{G}_k v}{v^T A v} \ , \tag{3.2}$$

$$\Gamma = \begin{pmatrix} 0 & \gamma_{01} & \cdots & & \gamma_{0J} \\ & 0 & \cdots & & \gamma_{1J} \\ & & \ddots & & \vdots \\ & & & 0 & \gamma_{(J-1)J} \\ & & & & 0 \end{pmatrix} , \tag{3.3}$$

where, for $k = 0, \dots, J-1$ and $l = k+1, \dots, J$,

$$\gamma_{kl} = \max_{w_k \in \mathbf{R}^{n_k}} \max_{v \in \mathbf{R}^n} \frac{v^T \check{G}_l^T (I - P_{l-1} G_{l-1})^T \check{P}_l^T A \check{P}_k w_k}{(w_k^T M_k w_k)^{1/2} (v^T \check{G}_l^T (I - P_{l-1} G_{l-1})^T M_l (I - P_{l-1} G_{l-1}) \check{G}_l v)^{1/2}} \ . \tag{3.4}$$

Then,

$$\rho(E_{MG}^{(J)}) \ \le \ 1 - \frac{2-\omega}{K(1 + \|\Gamma\|)^2} \tag{3.5}$$

where ω is defined by (2.3). Moreover,

$$\|\Gamma\| \ \le \ \frac{\omega}{\sqrt{2}} \sqrt{J(J+1)} \ . \tag{3.6}$$

Proof. We first gather some useful notation:

$$Q_k = (I - P_{k-1} G_{k-1}) \check{G}_k \qquad , \quad k = 0, \dots, J \ , \tag{3.7}$$
$$T_k = \check{P}_k (M_k)^{-1} \check{P}_k^T A \qquad , \quad k = 0, \dots, J \ , \tag{3.8}$$
$$F_k = (I - T_k)(I - T_{k-1}) \cdots (I - T_1)(I - T_0) \qquad , \quad k = 0, \dots, J \ . \tag{3.9}$$

In addition we set $F_{-1} = I$.

Following an idea in [24, Proposition 5.1.1], we note that

$$\begin{aligned}
& I - \check{P}_k (I_{n_k} - E_{MG}^{(k)}) A_k^{-1} \check{P}_k^T A \\
&= I - \check{P}_k \left(I_{n_k} - (I_{n_k} - M_k^{-1} A_k)(I_{n_k} - P_{k-1}(I_{n_{k-1}} - E_{MG}^{(k-1)}) A_{k-1}^{-1} P_{k-1} A_k)(I_{n_k} - M_k^{-1} A_k) \right) A_k^{-1} \check{P}_k^T A \\
&= I - \check{P}_k M_k^{-1} (2M_k - A_k) M_k^{-1} \check{P}_k^T - (I - \check{P}_k M_k^{-1} \check{P}_k^T A) \check{P}_{k-1} (I_{n_{k-1}} - E_{MG}^{(k-1)}) A_{k-1}^{-1} \check{P}_{k-1}^T A (I - \check{P}_k M_k^{-1} \check{P}_k^T A) \\
&= (I - T_k)(I - \check{P}_{k-1}(I_{n_{k-1}} - E_{MG}^{(k-1)}) A_{k-1}^{-1} \check{P}_{k-1}^T A)(I - T_k) \ .
\end{aligned}$$

Since $E_{MG}^{(J)} = I - \check{P}_J (I - E_{MG}^{(J)}) A^{-1} \check{P}_J^T A$ (because $\check{P}_J = I$), applying recursively this identity yields (using $E_{MG}^{(0)} = O$ and $M_0 = A_0$)

$$E_{MG}^{(J)} = (I - T_J)(I - T_{J-1}) \dots (I - T_1)(I - T_0)(I - T_1) \dots (I - T_{J-1})(I - T_J) \ .$$

Further, since $A^{-1}(I - T_k)^T = (I - T_k) A^{-1}$ and $(I - T_0)^2 = I - T_0$, one has $E_{MG}^{(J)} = F_J A^{-1} F_J^T A$, showing that

$$\rho(E_{MG}^{(J)}) = \rho(A^{1/2} E_{MG}^{(J)} A^{-1/2}) = \rho(A^{1/2} F_J A^{-1} F_J^T A^{1/2}) = \|F_J\|_A^2 = \max_{v \in \mathbf{R}^n} \frac{\|F_J v\|_A^2}{v^T A v} \ .$$

Using this relation, we first show that (3.5) holds if

$$v^T A v \le K \, (1 + \|\Gamma\|)^2 \ \left(\sum_{l=0}^{J} v^T F_{l-1}^T A T_l F_{l-1} v \right) \quad \forall v \in \mathbf{R}^n \ . \tag{3.10}$$

Indeed, since $AT_k = T_k^T A$, one has, $\forall v \in \mathbf{R}^n$,

$$\begin{aligned}
||F_{k-1}v||_A^2 - ||F_k v||_A^2 &= (F_{k-1}v)^T A F_{k-1} v - (F_{k-1}v)^T (I - T_k)^T A (I - T_k) F_{k-1} v \\
&= 2 v^T F_{k-1}^T A T_k F_{k-1} v - (F_{k-1}v)^T T_k^T A T_k (F_{k-1} v) \\
&= 2 v^T F_{k-1}^T A T_k F_{k-1} v - (F_{k-1}v)^T A \check{P}_k M_k^{-1} \check{P}_k^T A \check{P}_k M_k^{-1} \check{P}_k^T A (F_{k-1} v) \\
&= 2 v^T F_{k-1}^T A T_k F_{k-1} v - (F_{k-1}v)^T A \check{P}_k M_k^{-1} A_k M_k^{-1} \check{P}_k^T A (F_{k-1} v) \\
&\geq 2 v^T F_{k-1}^T A T_k F_{k-1} v - \omega \ (F_{k-1}v)^T A \check{P}_k M_k^{-1} \check{P}_k^T A (F_{k-1} v) \\
&= (2 - \omega) \ v^T F_{k-1}^T A T_k F_{k-1} v \ .
\end{aligned}$$

Summing both sides for $k = 0, \ldots, J$ shows that, $\forall v \in \mathbf{R}^n$,

$$\|v\|_A^2 - \|F_J v\|_A^2 \geq (2 - \omega) \left(\sum_{l=0}^{J} v^T F_{l-1}^T A T_l F_{l-1} v \right) \ ,$$

and it is straightforward to check that this relation, together with (3.10), implies (3.5), so that we are left with the proof of (3.10).

Now, observe that

$$\sum_{l=0}^{J} \check{P}_l Q_l \ = \ \sum_{l=0}^{J} \check{P}_l (I - P_{l-1} G_{l-1}) \check{G}_l \ = \ \sum_{l=0}^{J} \left(\check{P}_l \check{G}_l - \check{P}_{l-1} \check{G}_{l-1} \right) \ = \ \check{P}_J \check{G}_J - \check{P}_{-1} \check{G}_{-1} \ = \ I \ .$$

For any $v \in \mathbf{R}^n$, one may then decompose $v^T A v$ as the sum of two terms (remembering that $F_{-1} = I$):

$$v^T A v \ = \ \sum_{l=0}^{J} v^T A \check{P}_l Q_l v \ = \ \sum_{l=0}^{J} v^T F_{l-1}^T A \check{P}_l Q_l v \ + \ \sum_{l=1}^{J} v^T (I - F_{l-1}^T) A \check{P}_l Q_l v \ . \quad (3.11)$$

In order to prove (3.10), we bound separately the two terms in the right hand side of (3.11).

Regarding the first term, one has

$$\begin{aligned}
\sum_{l=0}^{J} v^T F_{l-1}^T A \check{P}_l Q_l v &\leq \sum_{l=0}^{J} (v^T F_{l-1}^T A \check{P}_l M_l^{-1} \check{P}_l^T A F_{l-1} v)^{1/2} (v^T Q_l^T M_l Q_l v)^{1/2} \\
&\leq \left(\sum_{l=0}^{J} v^T F_{l-1}^T A T_l F_{l-1} v \right)^{1/2} \left(\sum_{l=0}^{J} v^T Q_l^T M_l Q_l v \right)^{1/2} \qquad (3.12)
\end{aligned}$$

To estimate the second term, first observe that

$$I - F_{l-1} = I - (I - T_{l-1}) F_{l-2} = (I - F_{l-2}) + T_{l-1} F_{l-2} = \cdots = \sum_{k=0}^{l-1} T_k F_{k-1} \ .$$

Therefore,

$$\sum_{l=1}^{J} v^T(I - F_{l-1}^T)A\check{P}_l Q_l v = \sum_{l=1}^{J}\sum_{k=0}^{l-1} v^T F_{k-1}^T T_k^T A\check{P}_l Q_l v \ ,$$

whereas, for any $0 \le k < l \le J$, using (3.4) with $w_k = M_k^{-1}\check{P}_k^T A F_{k-1} v$,

$$\begin{aligned} v^T F_{k-1}^T T_k^T A\check{P}_l Q_l v &= (v^T F_{k-1}^T A\check{P}_k M_k^{-1})\check{P}_k^T A\check{P}_l Q_l v \\ &\le \gamma_{kl}(v^T Q_l^T M_l Q_l v)^{1/2}(v^T F_{k-1}^T A\check{P}_k M_k^{-1}\check{P}_k^T A F_{k-1} v)^{1/2} \\ &= \gamma_{kl}(v^T Q_l^T M_l Q_l v)^{1/2}(v^T F_{k-1}^T A T_k F_{k-1} v)^{1/2} \end{aligned}$$

Hence, since $\|\Gamma\| = \max_y \frac{\|\Gamma y\|}{\|y\|} = \max_{x,y}\frac{x^T\Gamma y}{\|x\|\,\|y\|}$,

$$\begin{aligned} \sum_{l=1}^{J} v^T(I - F_{l-1}^T)A\check{P}_l Q_l v &\le \sum_{l=1}^{J}\sum_{k=0}^{l-1} \gamma_{kl}(v^T Q_l^T M_l Q_l v)^{1/2}(v^T F_{k-1}^T A T_k F_{k-1} v)^{1/2} \\ &\le \|\Gamma\| \left(\sum_{l=0}^{J} v^T Q_l^T M_l Q_l v\right)^{1/2} \left(\sum_{k=0}^{J} v^T F_{k-1}^T A T_k F_{k-1} v\right)^{1/2} \end{aligned}$$

Combining this latter result with (3.12), one gets

$$v^T A v \ \le \ (1 + \|\Gamma\|) \left(\sum_{l=0}^{J} v^T Q_l^T M_l Q_l v\right)^{1/2} \left(\sum_{l=0}^{J} v^T F_{l-1}^T A T_l F_{l-1} v\right)^{1/2} .$$

Taking the square of both sides, and using (3.2) (which amounts to $\sum_{l=0}^{J} v^T Q_l^T M_l Q_l v \le K\, v^T A v$) straightforwardly leads to (3.10), which completes the proof of (3.5).

To prove (3.6), note that $\|\Gamma\| \le \|\Gamma\|_F = \left(\sum_{l=1}^{J}\sum_{k=0}^{l-1}\gamma_{kl}^2\right)^{1/2}$. Further, for any $0 \le k < l \le J$ and for any $w \in \mathbf{R}^n$ and $w_k \in \mathbf{R}^{n_k}$,

$$\begin{aligned} w^T Q_l^T \check{P}_l^T A\check{P}_k w_k &\le (w^T Q_l^T \check{P}_l^T A\check{P}_l Q_l w)^{1/2}(w_k^T \check{P}_k^T A\check{P}_k w_k)^{1/2} \\ &= (w^T Q_l^T A_l Q_l w)^{1/2}(w_k^T A_k w_k)^{1/2} \\ &\le \omega\,(w^T Q_l^T M_l Q_l w)^{1/2}\,(w_k^T M_k w_k)^{1/2} \ , \end{aligned}$$

showing that $\gamma_{kl} \le \omega$. The required result straightforwardly follows. ■

Note that (3.5) combined with (3.6) exhibits dependence on the number of levels. However, as shown in Theorem 3.4 below, there are choices of the pseudo restrictions G_k for which a further analysis of $\|\Gamma\|$ is possible, opening the way to bounds that are optimal with respect to the number of levels.

Now, before discussing how to select G_k in Theorem 3.1, let's consider the influence of the number of pre- and post-smoothing steps ν_k. The following theorem brings some light in this respect.

Theorem 3.2 *Let the assumptions of Theorem 3.1 hold. Assume, in addition that* $\nu_k = \nu$ *is the same for* $k = 1, \ldots, J$ *, and such that* $\nu > 1$ *. Let* ω*,* ω_1 *be defined by (2.3), (2.4) respectively. Let* $R_0 = A_0$ *and*

$$K_1 = \max_{v \in \mathbf{R}^n} \frac{\sum_{k=0}^{J} \; v^T \check{G}_k^T (I - P_{k-1} G_{k-1})^T R_k (I - P_{k-1} G_{k-1}) \check{G}_k v}{v^T A v} \; . \tag{3.13}$$

Then,

$$\omega \;\leq\; \widetilde{\omega} \; , \tag{3.14}$$

$$K \;\leq\; \alpha^{-1} K_1 \; , \tag{3.15}$$

where K *is defined as in Theorem 3.1, and where*

$$\widetilde{\omega} = \begin{cases} 1 & \textit{if } \nu \textit{ is even} \\ 1 + (\omega_1 - 1)^\nu & \textit{otherwise} \; , \end{cases} \tag{3.16}$$

$$\alpha = \begin{cases} \frac{1 - (\omega_1 - 1)^\nu}{\omega_1} & \textit{if } \nu \textit{ is even} \\ \frac{1}{\omega_1} & \textit{otherwise} \; . \end{cases} \tag{3.17}$$

Moreover,

$$\rho(E_{MG}^{(J)}) \;\leq\; 1 - \frac{1 - (\omega_1 - 1)^\nu}{\omega_1 \, K_1 (1 + \|\Gamma\|)^2} \;\leq\; 1 - \frac{2 - \omega_1}{K_1 (1 + \|\Gamma\|)^2} \; , \tag{3.18}$$

Proof. One has $\omega = \max\left(1 \, , \, \max_k \lambda_{\max}(M_k^{-1} A_k)\right)$ (where $\lambda_{\max}(\cdot)$ denotes the largest eigenvalue) with

$$\lambda_{\max}(M_k^{-1} A_k) = \lambda_{\max}\left(I - (I - R_k^{-1} A_k)^\nu\right) = \max_{\lambda \in \sigma(R_k^{-1} A_k)} 1 - (1 - \lambda)^\nu \; .$$

From there, it is readily seen that $\widetilde{\omega}$ defined by (3.16) is an upper bound on ω.

On the other hand, $K \leq \alpha^{-1} K_1$ holds if, for $1 \leq k \leq J$,

$$v^T M_k v \leq \alpha^{-1} \, v^T R_k v \quad \forall v \in \mathbf{R}^{n_k} \; ,$$

and the latter relation is true if and only if, $\forall \lambda \in \sigma(R_k^{-1} A_k)$,

$$1 - (1 - \lambda)^\nu \geq \alpha \, \lambda \; . \tag{3.19}$$

This relation holds for $\alpha = 1$ when $\lambda \leq 1$ (since then $1 - \lambda \geq (1 - \lambda)^\nu$). When ν is even, $(1 - (\lambda - 1)^\nu) / \lambda$ is a decreasing function of λ for $1 < \lambda \leq \omega_1 < 2$, thus not smaller than $(1 - (\omega_1 - 1)^\nu) / \omega_1$. On the other hand, when $\lambda > 1$ and ν is odd, (3.19) holds for $\alpha = 1/\omega_1$ since $1 + (\lambda - 1)^\nu \geq 1 \geq \lambda/\omega_1$. The conclusion follows because, for $\nu > 1$, $\frac{1-(\omega_1-1)^\nu}{\omega_1} \geq \frac{1-(\omega_1-1)^2}{\omega_1} = 2 - \omega_1$. ■

The bound (3.18) improves when the number of smoothing steps increases. However, the improvement is slight, and there is no improvement if $\nu = 2$ or $\omega_1 = 1$. Therefore Theorem 3.2 essentially shows that the estimates derived for the case $\nu = 1$ may serve as worst case estimate for the general case, but does not allow to really assess the effect of an increase of the number of smoothing steps. In the next section, we show on an example that this effect is better seen by computing directly the constant K with M_k defined from (2.2).

Analysis in generalized hierarchical basis

Our analysis holds for any G_k of appropriate dimensions. However, looking at the numerators of (3.2), (3.13), there is some advantage in setting G_k in such a way that P_kG_k is a projector: then, each term of the sum involves M_k restricted to a particular subspace. Note that P_kG_k is a projector when $G_kP_k = I_{n_k}$.

Now, often P_k has the form

$$P_k = \begin{pmatrix} J_k \\ I_{n_k} \end{pmatrix} \tag{3.20}$$

where J_k is a $(n_{k+1} - n_k) \times n_k$ "interpolation" matrix. Then, an obvious choice for G_k is

$$G_k = \begin{pmatrix} O_{n_k \times (n_{k+1}-n_k)} & I_{n_k} \end{pmatrix} , \tag{3.21}$$

which further gives

$$\check{G}_k = \begin{pmatrix} O_{n_k \times (n-n_k)} & I_{n_k} \end{pmatrix} .$$

With this form of P_k and this choice of G_k, it is interesting to introduce the (generalized) hierarchical basis associated to the transformation

$$S_k = \begin{pmatrix} I_{n_{k+1}-n_k} & J_k \\ & I_{n_k} \end{pmatrix} . \tag{3.22}$$

For finite element problems with regular refinement, and assuming that J_k in (3.20) is the standard interpolation matrix, this corresponds to the usual transformation that brings a vector defined in the hierarchical basis at level $k+1$ to the nodal basis. When there are more than two levels, a global basis transformation is obtained with

$$\check{S}_k = S_k \begin{pmatrix} I_{n_{k+1}-n_k} & \\ & \check{S}_{k-1} \end{pmatrix} , \quad k = 1, \ldots, J-1 . \tag{3.23}$$

Outside the finite element context, (3.22) defines a *generalized* hierarchical basis transformation as introduced in [7], and (3.23) may be seen as the proper multilevel extension of this concept.

The matrix A in (generalized) hierarchical basis is

$$\hat{A} = \check{S}_{J-1}^T A \check{S}_{J-1} \ , \tag{3.24}$$

and we would like to analyze the constant K in this basis; that is, to analyze

$$K = \max_{v \in \mathbf{R^n}} \frac{\left(\sum_{k=0}^{J} v^T \check{S}_{J-1}^T \check{G}_k^T (I - P_{k-1} G_{k-1})^T M_k (I - P_{k-1} G_{k-1}) \check{G}_k \check{S}_{J-1} v \right)}{v^T \hat{A} v} \ .$$

In this view, we first observe that G_k is unchanged under hierarchical basis transformation:

$$\begin{aligned} \check{S}_{k-1}^{-1} G_k \check{S}_k &= \check{S}_{k-1}^{-1} \begin{pmatrix} O_{n_k \times (n_{k+1} - n_k)} & I_{n_k} \end{pmatrix} \begin{pmatrix} I_{n_{k+1}-n_k} & \\ & \check{S}_{k-1} \end{pmatrix} \\ &= \begin{pmatrix} O_{n_k \times (n_{k+1} - n_k)} & I_{n_k} \end{pmatrix} \ . \end{aligned}$$

Therefore,

$$\check{S}_{k-1}^{-1} \check{G}_k \check{S}_{J-1} = \check{S}_{k-1}^{-1} G_k \check{S}_k \check{S}_k^{-1} G_{k+1} \check{S}_{k+1} \cdots \check{S}_{J-2}^{-1} G_{J-1} \check{S}_{J-1} = G_k G_{k+1} \cdots G_{J-1} = \check{G}_k$$

and

$$\begin{aligned} (I - &P_{k-1} G_{k-1}) \check{G}_k \check{S}_{J-1} \\ &= (I - P_{k-1} G_{k-1}) \check{S}_{k-1} \check{G}_k \\ &= \begin{pmatrix} I_{n_k - n_{k-1}} & -J_{k-1} \\ & O_{n_{k-1} \times n_{k-1}} \end{pmatrix} \begin{pmatrix} I_{n_k - n_{k-1}} & J_k \\ & I_{n_{k-1}} \end{pmatrix} \begin{pmatrix} I_{n_k - n_{k-1}} & \\ & \check{S}_{k-2} \end{pmatrix} \check{G}_k \\ &= \begin{pmatrix} I_{n_k - n_{k-1}} & \\ & O_{n_{k-1} \times n_{k-1}} \end{pmatrix} \begin{pmatrix} O_{(n_k - n_{k-1}) \times (n - n_k)} & I_{n_k - n_{k-1}} & \\ O_{n_{k-1} \times (n - n_k)} & & \check{S}_{k-2} \end{pmatrix} \\ &= \begin{pmatrix} O_{(n_k - n_{k-1}) \times (n - n_k)} & I_{n_k - n_{k-1}} & \\ O_{n_{k-1} \times (n - n_k)} & & O_{n_{k-1} \times n_{k-1}} \end{pmatrix} \ . \end{aligned} \tag{3.25}$$

Here, it is interesting to note that, for $k = 1, \ldots, J$, the prolongation P_{k-1} of the form (3.20) induces a partitioning of unknowns at level k into "fine grid unknowns" (the first block of size $n_k - n_{k-1}$) and "coarse grid unknowns" (the remaining block of size n_{k-1}). Accordingly, M_k and R_k may be partitioned in 2×2 block form

$$M_k = \begin{pmatrix} M_k^{(FF)} & M_k^{(FC)} \\ M_k^{(CF)} & M_k^{(CC)} \end{pmatrix} \quad , \quad R_k = \begin{pmatrix} R_k^{(FF)} & R_k^{(FC)} \\ R_k^{(CF)} & R_k^{(CC)} \end{pmatrix} \ , \tag{3.26}$$

and we may define the $n \times n$ block diagonal matrices which concatenate the top left block of these matrices:

$$D_M = \begin{pmatrix} M_J^{(FF)} & & & \\ & \ddots & & \\ & & M_1^{(FF)} & \\ & & & A_0 \end{pmatrix} \quad , \quad D_R = \begin{pmatrix} R_J^{(FF)} & & & \\ & \ddots & & \\ & & R_1^{(FF)} & \\ & & & A_0 \end{pmatrix} . \tag{3.27}$$

With (3.25), it is straightforward to check the following theorem.

Theorem 3.3 *Let the assumptions of Theorem 3.1 hold. Assume in addition that* P_k , $k = 0, \dots, J-1$, *has the form* (3.20) *and let* G_k , $k = 0, \dots, J-1$, *be defined by* (3.21). *Let* $\hat{A}$ *be defined by* (3.24), *where* $\check{S}_{J-1}$ *is defined from* (3.22), (3.23), *and let* D_M , D_R *be defined by* (3.26), (3.27).

Then:

$$K = \max_{v \in \mathbf{R^n}} \frac{v^T D_M v}{v^T \hat{A} v} \quad , \quad K_1 = \max_{v \in \mathbf{R^n}} \frac{v^T D_R v}{v^T \hat{A} v} \tag{3.28}$$

where K *and* K_1 *are defined by, respectively,* (3.2) *and* (3.13).

Several works address the conditioning of finite element matrices in the hierarchical basis, sometimes with respect to some well-conditioned block diagonal matrix [1, 2, 28, 29]. The main goal is a proof that the so-called hierarchical basis multigrid method has near optimal convergence. In this latter method, only fine grid unknowns are relaxed during smoothing steps, which makes the connection with the conditioning in hierarchical basis straightforward [23]. With Theorem 3.3, one sees that the good conditioning of the matrix in the hierarchical basis also proves the near optimality of standard multigrid methods. This extends to more than two levels one of the conclusions in [16]: when the fast convergence of the hierarchical basis multigrid method can be proved, standard multigrid methods also converge fast, so that the former cannot have, from a theoretical viewpoint, a decisive advantage over the latter.

A-orthogonal projection

A more powerful and more general choice than (3.21) is obtained with

$$G_k = A_k^{-1} P_k^T A_{k+1} \ . \tag{3.29}$$

With this choice, $P_k G_k$ is a A_{k+1}–orthogonal projector: $A_{k+1} P_k G_k = (P_k G_k)^T A_{k+1}$. One has also

$$\check{G}_k = A_k^{-1} \check{P}_k^T A \ , \tag{3.30}$$

and $\check{P}_k \check{G}_k$ is a A–orthogonal projector.

Theorem 3.4 *Let the assumptions of Theorem 3.1 hold, and let* G_k *,* $k = 0, \dots, J-1$ *, be defined by* (3.29). *Then,* K *,* Γ *, defined as in Theorem 3.1, and* K_1 *, defined as in Theorem 3.2, satisfy*

$$K = \max\left(1\,,\ \max_{1\le k\le J}\ \max_{w_k\in\mathbf{R}^{n_k}}\ \frac{w_k^T(I-\pi_{A_k})^T M_k (I-\pi_{A_k})w_k}{w_k^T(I-\pi_{A_k})^T A_k (I-\pi_{A_k})w_k}\right) \tag{3.31}$$

$$= \max\left(1\,,\ \max_{1\le k\le J}\ \max_{w_k\in\mathbf{R}^{n_k}}\ \frac{w_k^T(I-\pi_{A_k})^T M_k (I-\pi_{A_k})w_k}{w_k^T A_k w_k}\right)\ , \tag{3.32}$$

$$K_1 = \max\left(1\,,\ \max_{1\le k\le J}\ \max_{w_k\in\mathbf{R}^{n_k}}\ \frac{w_k^T(I-\pi_{A_k})^T R_k (I-\pi_{A_k})w_k}{w_k^T(I-\pi_{A_k})^T A_k (I-\pi_{A_k})w_k}\right) \tag{3.33}$$

$$= \max\left(1\,,\ \max_{1\le k\le J}\ \max_{w_k\in\mathbf{R}^{n_k}}\ \frac{w_k^T(I-\pi_{A_k})^T R_k (I-\pi_{A_k})w_k}{w_k^T A_k w_k}\right)\ , \tag{3.34}$$

$$\Gamma = 0\ , \tag{3.35}$$

where

$$\pi_{A_k} = P_{k-1}G_{k-1} = P_{k-1}A_{k-1}^{-1}P_{k-1}^T A_k\ . \tag{3.36}$$

Proof. We first prove (3.35). Using (3.30), one has, for any $0 \le k < l \le J$ and all $w_k \in \mathbf{R}^{n_k}$, $v \in \mathbf{R}^n$,

$$\begin{aligned} w_k^T \check{P}_k^T A \check{P}_l (I - P_{l-1}G_{l-1})\check{G}_l v &= w_k^T \check{P}_k^T A \check{P}_l A_l^{-1} \check{P}_l^T A v - w_k^T \check{P}_k^T A \check{P}_{l-1} A_{l-1}^{-1} \check{P}_{l-1}^T A v \\ &= w_k^T P_k^T \cdots P_{l-1}^T \left(\check{P}_l^T A \check{P}_l A_l^{-1}\right) \check{P}_l^T A v \\ &\quad - w_k^T P_k^T \cdots P_{l-2}^T \left(\check{P}_{l-1}^T A \check{P}_{l-1} A_{l-1}^{-1}\right) \check{P}_{l-1}^T A v \\ &= w_k^T P_k^T \cdots P_{l-1}^T \check{P}_l^T A v - w_k^T P_k^T \cdots P_{l-2}^T \check{P}_{l-1}^T A v \\ &= w_k^T \check{P}_k^T A v - w_k^T \check{P}_k^T A v \\ &= 0\ ; \end{aligned}$$

$\gamma_{kl} = 0$ and therefore $\Gamma = 0$ readily follows.

We next prove (3.31), (3.32). The proof of (3.33), (3.34) is similar and will be omitted. We first show that

$$\sum_{k=0}^{J} \check{G}_k^T (I - P_{k-1}G_{k-1})^T A_k (I - P_{k-1}G_{k-1})\check{G}_k \ = \ A\ . \tag{3.37}$$

Indeed, using (3.30), one has, for $k = 1, \dots, J$:

$$\begin{aligned} &\check{G}_k^T (I-\pi_{A_k})^T A_k (I-\pi_{A_k})\check{G}_k \\ &= (A_k^{-1}\check{P}_k^T A - P_{k-1}A_{k-1}^{-1}\check{P}_{k-1}^T A)^T A_k (A_k^{-1}\check{P}_k^T A - P_{k-1}A_{k-1}^{-1}\check{P}_{k-1}^T A) \\ &= A(\check{P}_k A_k^{-1} - \check{P}_{k-1}A_{k-1}^{-1}P_{k-1})(\check{P}_k^T - A_k P_{k-1}A_{k-1}^{-1}\check{P}_{k-1}^T)A \\ &= A(\check{P}_k A_k^{-1}\check{P}_k^T - \check{P}_{k-1}A_{k-1}^{-1}\check{P}_{k-1}^T)A\ , \end{aligned}$$

hence

$$\sum_{k=0}^{J} \check{G}_k^T (I - P_{k-1}G_{k-1})^T A_k (I - P_{k-1}G_{k-1})\check{G}_k$$

$$\begin{aligned}
&= \sum_{k=1}^{J} A(\check{P}_k A_k^{-1}\check{P}_k^T - \check{P}_{k-1}A_{k-1}^{-1}\check{P}_{k-1}^T)A + \check{G}_0^T A_0 \check{G}_0 \\
&= A\left(\sum_{k=1}^{J}\left(\check{P}_k A_k^{-1}\check{P}_k^T - \check{P}_{k-1}A_{k-1}^{-1}\check{P}_{k-1}^T\right) + \check{P}_0 A_0^{-1}\check{P}_0^T\right) A \\
&= A\ .
\end{aligned}$$

Then,

$$\begin{aligned}
K &= \max_{v\in\mathbf{R}^n} \frac{\sum_{k=0}^{J}\ v^T\check{G}_k^T(I-P_{k-1}G_{k-1})^T M_k (I-P_{k-1}G_{k-1})\check{G}_k v}{\sum_{k=0}^{J}\ v^T\check{G}_k^T(I-P_{k-1}G_{k-1})^T A_k (I-P_{k-1}G_{k-1})\check{G}_k v} \qquad (3.38)\\
&= \max_{v\in\mathbf{R}^n} \frac{\sum_{k=1}^{J}\ v^T\check{G}_k^T(I-\pi_{A_k})^T M_k (I-\pi_{A_k})\check{G}_k v\ +\ v^T\check{G}_0^T A_0\check{G}_0 v}{\sum_{k=1}^{J}\ v^T\check{G}_k^T(I-\pi_{A_k})^T A_k (I-\pi_{A_k})\check{G}_k v\ +\ v^T\check{G}_0^T A_0\check{G}_0 v} \\
&\le \max\left(1\,,\ \max_{1\le k\le J}\ \max_{w_k\in\mathbf{R}^{n_k}} \frac{w_k^T(I-\pi_{A_k})^T M_k (I-\pi_{A_k})w_k}{w_k^T(I-\pi_{A_k})^T A_k (I-\pi_{A_k})w_k}\right)\ .
\end{aligned}$$

This proves that the right hand side of (3.31) is an upper bound on K ; the right hand side of (3.32) is a further upper bound since

$$\max_{w_k\in\mathbf{R}^{n_k}} \frac{w_k^T(I-\pi_{A_k})^T R_k (I-\pi_{A_k})w_k}{w_k^T A_k w_k} \ \ge\ \max_{v_k\in\mathbf{R}^{n_k}} \frac{v_k^T(I-\pi_{A_k})^T R_k (I-\pi_{A_k})v_k}{v_k^T(I-\pi_{A_k})^T A_k (I-\pi_{A_k})v_k}\ ,$$

as seen by restricting the maximum in the left hand side to $w_k = (I-\pi_{A_k})v_k$ (taking into account that $(I-\pi_{A_k})^2 = (I-\pi_{A_k})$).

To prove that the right hand sides of (3.31), (3.32) are also a lower bound on K , let, for $k = 0,\dots,J$, $Q_k = (I - P_{k-1}G_{k-1})\check{G}_k$, then rewrite (3.38) as

$$K = \max_{v\in\mathbf{R}^\mathbf{n}} \frac{\sum_{k=0}^{J} v^T Q_k^T M_k Q_k v}{\sum_{k=0}^{J} v^T Q_k^T A_k Q_k v} \qquad (3.39)$$

Since $G_k P_k = I_{n_k}$ for $k = 0,\dots,J-1$, Lemma A.1 in appendix proves that, for $0 \le l, k \le J$ with $k \ne l$,

$$Q_l \check{P}_l Q_l = Q_l \quad \text{and} \quad Q_k \check{P}_l Q_l = O_{n_k\times n}\ .$$

Restricting the maximum in (3.39) to $v = \check{P}_l Q_l w$ for some $0 \le l \le J$ yields

$$\begin{aligned}
K &\geq \max_{w\in\mathbf{R}^n} \frac{w^T Q_l^T M_l Q_l w}{w^T Q_l^T A_l Q_l w} \\
&= \max_{w\in\mathbf{R}^n} \frac{w^T \check{G}_l^T (I - P_{l-1}G_{l-1})^T M_l (I - P_{l-1}G_{l-1})\check{G}_l w}{w^T \check{G}_l^T (I - P_{l-1}G_{l-1})^T A_l (I - P_{l-1}G_{l-1})\check{G}_l w} \\
&= \max_{w_l\in\mathbf{R}^{n_l}} \frac{w_l^T (I - P_{l-1}G_{l-1})^T M_l (I - P_{l-1}G_{l-1}) w_l}{w_l^T (I - P_{l-1}G_{l-1})^T A_l (I - P_{l-1}G_{l-1}) w_l} ,
\end{aligned}$$

the last equality stemming from the fact that $\check{G}_l$ has full rank (from (3.30), and because $\check{P}_k$ has full rank by virtue of our general assumptions). The conclusion follows because

$$\begin{aligned}
w_l^T (I - P_{l-1}G_{l-1})^T A_l (I - P_{l-1}G_{l-1}) w_l &= w_l^T (I - \pi_{A_l})^T A_l (I - \pi_{A_l}) w_l \\
&= w_l^T (A_l - A_l P_{l-1} A_{l-1}^{-1} P_{l-1}^T A_l) w_l \\
&\leq w_l^T A_l w_l .
\end{aligned}$$

■

Comparison with the original SSC theory

Theorem 3.1 resembles its counterparts in [26, 30]. However, a further comparative discussion is difficult because the way we exploit this result via Theorems 3.3 and 3.4 is different in spirit from the way the analysis is conducted in [26, 30].

In [26, 30], the focus is on the proof that $K = \mathcal{O}(1)$ for a class of finite element problems. No attempt is made to derive expressions that could be assessed in concrete examples, despite some similarities between the choice (3.29) for G_k and the so-called a–orthogonal subspace decomposition used in [30]. Moreover, the constant which plays the role of $\|\Gamma\|$ in our analysis is also not analyzed as accurately. In [26], it is just stated that this constant does not exceed ωJ , which is less sharp than (3.6). In [30], it is indeed observed that $\gamma_{kl} = 0$ for $k < l$ when using the a–orthogonal subspace decomposition, but the constant appearing in the bound also involves γ_{kl} for $k = l$, and the reader is left with the argument that this constant is $\mathcal{O}(1)$, without further comments.

On the other hand, with our algebraic approach, the way towards a (near) optimal bound on the convergence factor can also be straightforward. Regarding Theorem 3.3, this is discussed above in connection with the conditioning in the hierarchical basis. On the other hand, K as given in Theorem 3.4 is the maximum over all levels of an expression which involves only two subsequent levels. Any analysis proving a bound for this expression that is independent of the grid size shows therefore $\rho(E_{MG}^{(J)}) \leq 1-(2-\omega)/K = \mathcal{O}(1)$. However, our approach is not limited to qualitative results. We first develop an accurate analysis of $\|\Gamma\|$, given either by the general bound (3.6), or by the particular result in Theorem 3.4. Furthermore, as illustrated in the next section, K as given in Theorem 3.4 may be numerically and/or analytically assessed in concrete examples, yielding a satisfactorily sharp prediction of actual multigrid convergence.

V-cycle versus two-grid convergence analysis

One may observe some similarities between the bound resulting from Theorems 3.1 and 3.4 and the two-grid convergence estimates derived in [8, 9]. The results in [8] involve a freely chosen projector Q onto the range of P (this freedom is somehow comparable to that of choosing G_k in Theorem 3.1). Now, for two levels ($J = 1$), our analysis re-gives the bound obtained by combining Theorem 2.2 and Lemma 2.3 in [8], for the special case $Q = \pi_A$. For this case we therefore prove that the bound on the two-grid convergence rate also applies to V-cycle multigrid.

However, there are multigrid methods which scale poorly with V-cycle but have nice convergence properties when only two levels are used. This, in particular, holds for (non-smoothed) aggregation-based multigrid methods [15]. In such cases, the analysis of the constant K in Theorem 3.4 has to reveal a dependence with the problem size. On the other hand, as shown in [9], there are choices of the projector Q for which the analysis in [8] is sharp and therefore has to show the grid-independent convergence of the two-grid variant. Hence there are subtle but essential differences between our results and the two-grid convergence results in [8, 9]. A further analysis should then help to understand the conditions under which an optimal two-grid method remains optimal when used recursively in a V-cycle scheme – a question unanswered so far. Its discussion lies, however, outside the scope of the present paper.

4 Example

We consider the linear system resulting from the bilinear finite element discretization of

$$
\begin{aligned}
-\Delta\, u &= f \quad \text{in} \quad \Omega = (0,1)\times(0,1) \\
u &= 0 \quad \text{in} \quad \partial\Omega
\end{aligned}
$$

on a uniform grid of mesh size $h = 1/N_J$ in both directions. This leads to the following nine point stencil

$$\begin{bmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{bmatrix} , \tag{4.1}$$

which also corresponds, up to some scaling factor, to the 9-point finite difference discretization.

We assume $N_J = 2^J N_0$ for some integer N_0, allowing J steps of regular geometric coarsening. We consider prolongations of the form (3.20), where J_k is the standard interpolation associated with bilinear finite element basis functions. The restriction P_k^T corresponds then to "full weighting", as defined in, e.g. [22] [1]. With these choices, the stencil (4.1) is preserved throughout all grids (up to some unimportant scaling factor), and K as given in Theorem 3.4 may be assessed by analyzing

$$\max_{w_k} \frac{w_k^T (I - \pi_{A_k})^T M_k (I - \pi_{A_k}) w_k}{w_k A_k w_k} \tag{4.2}$$

for a matrix A_k corresponding to stencil (4.1) applied on a grid with mesh size $h_k = 1/N_k$. Considering two successive grids is therefore sufficient, and, to alleviate notation, we let $N = N_k$, $A = A_k$, $M = M_k$, $P = P_{k-1}$, $A_c = A_{k-1} = P^T A P$ and $\pi_A = \pi_{A_k} = P A_c^{-1} P^T A$.

To assess (4.2), we resort to Fourier analysis. The eigenvectors of A are, for $m, l = 1, \ldots, N-1$, the functions

$$u_{m,l}^{(N)} = \sin(m\pi x) \sin(l\pi y)$$

evaluated at the grid points. The eigenvalue corresponding to $u_{m,l}^{(N)}$ is

$$\lambda_{m,l}^{(N)} = 4(3 s_m + 3 s_l - 4 s_m s_l) \tag{4.3}$$

where

$$s_m = \sin^2(m\pi/2N) \quad , \quad s_l = \sin^2(l\pi/2N) \ . \tag{4.4}$$

The prolongation P satisfies (see, e.g., [22, p. 87])

$$P^T \left\{ \begin{array}{c} u_{m,l}^{(N)} \\ u_{N-m,N-l}^{(N)} \\ -u_{N-m,l}^{(N)} \\ -u_{m,N-l}^{(N)} \end{array} \right\} = 4 \left\{ \begin{array}{c} (1-s_m)(1-s_l) \\ s_m s_l \\ s_m(1-s_l) \\ (1-s_m) s_l \end{array} \right\} u_{m,l}^{(N/2)}$$

[1] up to some scaling factor; the scalings of the prolongation and restriction are unimportant when using coarse grid matrices of the Galerkin type.

for $1 \leq m,l \leq N/2-1$, with $P^T u_{m,l}^{(N)} = 0$ for $m = N/2$ or $m = N/2$. Expressed in the Fourier basis (that is, in the basis of eigenvectors of A), $I - \pi_A$ is therefore block diagonal with, for $1 \leq m,l \leq N/2-1$, 4×4 blocks

$$(I - \pi_A)_{m,l} = I_4 - P_{m,l} \left(A_{m,l}^{(c)} \right)^{-1} P_{m,l}^T A_{m,l} \tag{4.5}$$

where

$$\begin{aligned}
P_{m,l}^T &= 4 \begin{pmatrix} (1-s_m)(1-s_l) & s_m s_l & s_m(1-s_l) & (1-s_m)s_l \end{pmatrix} \\
A_{m,l} &= \operatorname{diag}\left(\lambda_{m,l}^{(N)} , \lambda_{N-m,N-l}^{(N)} , \lambda_{m,N-l}^{(N)} , \lambda_{N-m,l}^{(N)} \right) \\
A_{m,l}^{(c)} &= P_{m,l}^T A_{m,l} P_{m,l} = 64\big(3s_m(1-s_m) + 3s_l(1-s_l) - 16 s_l(1-s_l)s_m(1-s_m)\big) .
\end{aligned}$$

For $m = N/2$, $1 \leq l \leq N/2-1$ and $l = N/2$, $1 \leq m \leq N/2-1$, $(I - \pi_A)_{m,l} = I_2$ is a 2×2 identity block, whereas $(I - \pi_A)_{\frac{N}{2}\frac{N}{2}} = 1$ reduces to the scalar identity. If M in the Fourier basis has the same block diagonal structure, we are left with the analysis of

$$\rho_{m,l} = \rho\left((I - \pi_A)_{m,l}^T M_{m,l} (I - \pi_A)_{m,l} A_{m,l}^{-1} \right) . \tag{4.6}$$

Now, we consider more specifically damped Jacobi smoothing with $\omega_{\text{Jac}} = 0.5$; that is $R_k = R = 2\text{diag}(A) = 16\,I$. Then, for any number of pre- and post-smoothing steps $\nu_k = \nu$, M is diagonal in the Fourier basis, with diagonal entries depending on the eigenvalues of A; that is (see (4.3)), depending on s_m and s_l.To obtain grid independent bounds, it is then interesting to consider $\rho_{m,l} = \rho(s_m, s_l)$ (with $(I - \pi_A)_{m,l}$ as in (4.5)) as a function of s_m, s_l, and let these parameters vary continuously in $[0,1]$, excluding the corner points where $s_m(1-s_m) = s_l(1-s_l) = 0$, which correspond to singularities. For both $\nu = 1$ and $\nu = 2$, $\rho(s_m, s_l)$ presents the following symmetries: $\rho(s_m, s_l) = \rho(s_l, s_m) = \rho(1-s_m, s_s) = \rho(s_m, 1-s_l) = \rho(1-s_m, 1-s_l)$. Further, numerical investigations reveal that the maximum on the considered domain is located at the boundary, i.e., corresponds to, e.g., $s_m = 0$. Because of the symmetries it is sufficient to analyze this latter case. One may check that $\rho(0, s_l)$ is the largest eigenvalue in modulus of

$$\frac{1}{4} \begin{pmatrix} \frac{s_l\mu_1 + s_l\mu_4}{3} & 0 & 0 & -\frac{s_l\mu_1 + s_l\mu_4}{3} \\ 0 & \frac{\mu_2}{3-(1-s_l)} & 0 & 0 \\ 0 & 0 & \frac{\mu_3}{3-s_l} & 0 \\ -\frac{\mu_1(1-s_l) + \mu_4(1-s_l)}{3} & 0 & 0 & \frac{(1-s_l)\mu_1 + (1-s_l)\mu_4}{3} \end{pmatrix} ,$$

where $\{\mu_i\}_{i=1,\dots,4}$ are the 4 diagonal entries of M_{kl}. Thus:

$$\rho(0, s_l) = \max\left(\frac{\mu_3}{3-s_l} , \frac{\mu_2}{3-(1-s_l)} , \frac{\mu_1 + \mu_4}{3} \right) .$$

For $\nu = 1$, $\mu_i = 16$ for all i, hence $\rho(0, s_l) \le 8/3$. Inspecting 2×2 blocks and the 1×1 remaining block reveals that $8/3$ is also an upper bound on $\rho_{m,l}$ when $m = N/2$ or $l = N/2$. Hence we have

$$K \le \frac{8}{3} ,$$

showing that, since $\omega = 1$ for this smoother,

$$\rho(E_{MG}^{(J)}) \ \le \ 1 - \frac{1}{K} \ \le \ 0.625$$

($\|\Gamma\| = 0$ because we are in the framework of Theorem 3.4).

We also used Fourier analysis to compute the convergence factor of the corresponding two-grid method, which is $\sigma_2 \approx 0.391$. This is also a lower bound on the V-cycle convergence factor, and one sees that our upper bound above is not far from this lower bound. On the other hand, the standard bound on the W-cycle convergence factor is $\sigma_2/(1-\sigma_2) \approx 0.642$ [17]. This is less favorable than our bound for the V-cycle. Since the W-cycle convergence factor cannot be worse than the V-cycle convergence factor, our bound gives also, in this case, a slightly better estimate for the W-cycle.

For $\nu = 2$,

$$M \ = \ R(2R - A)^{-1}R \ = \ 256(32I - A)^{-1} ,$$

entailing that, for $s_m = 0$, $\mu_1 = 64/(8 - 3s_l)$, $\mu_2 = 64/(8 - (2 + s_l))$, $\mu_3 = 64/(8 - (3 - s_l))$ and $\mu_4 = 64/(8 - 3(1 - s_l))$. Hence,

$$\rho(0, s_l) = \max \left[\frac{16}{(5 + s_l)(3 - s_l)} , \frac{16}{(6 - s_l)(2 + s_l)} , \frac{16}{3} \left(\frac{1}{8 - 3s_l} + \frac{1}{5 + 3s_l} \right) \right] .$$

The last term is maximum for $s_l = 0$ or $s_l = 1$, yielding $\rho(0, s_l) \le 26/15$. Here again, inspecting 2×2 blocks and the 1×1 remaining block reveals that $26/15$ is also an upper bound on $\rho_{m,l}$ when $m = N/2$ or $l = N/2$. Hence

$$K \le \frac{26}{15} ,$$

showing that,

$$\rho(E_{MG}^{(J)}) \ \le \ 1 - \frac{1}{K} \ \le \ 0.423 .$$

Theorem 3.2 gives thus only a worst case estimate, whereas computing directly K with M_k defined from (2.2) allows to prove better convergence factors when the number of smoothing steps increases.

Appendix A

Lemma A.1 *Let* P_k , $k = 0, \ldots, J-1$ *be* $n_{k+1} \times n_k$ *matrices of rank* n_k *with* $n = n_J > n_{J-1} > \cdots > n_0$. *Let* G_k , $k = 0, \ldots, J-1$ *be* $n_{k+1} \times n_k$ *matrices such that*

$$G_k\, P_k = I_{n_k} \ .$$

Set $P_{-1} = G_{-1} = O_{n_0 \times n_0}$ *and let, for* $k = 0, \ldots, J$, $\check{P}_k$ *be defined by* (2.5), $\check{G}_k$ *be defined by* (3.1), *and* $Q_k = (I - P_{k-1}G_{k-1})\check{G}_k$.

There holds, for $0 \leq l, k \leq J$ *with* $k \neq l$,

$$Q_k \check{P}_k Q_k = Q_k \qquad \text{and} \qquad Q_l \check{P}_k Q_k = O_{n_l \times n} \ .$$

Proof. Note that $G_k\, P_k = I_{n_k}$ implies $\check{G}_k \check{P}_k = I_{n_k}$. The first statement follows then from

$$\begin{aligned}(I - P_{k-1}G_{k-1})\check{G}_k\check{P}_k(I - P_{k-1}G_{k-1}) &= (I - P_{k-1}G_{k-1})(I - P_{k-1}G_{k-1}) \\ &= I - P_{k-1}G_{k-1} \ .\end{aligned}$$

To prove the second statement, we consider two cases. If $l > k$,

$$\begin{aligned}(I - P_{l-1}G_{l-1})\check{G}_l\check{P}_k &= (I - P_{l-1}G_{l-1})G_l \cdots G_{J-1}P_{J-1} \cdots P_l P_{l-1} \cdots P_k \\ &= (I - P_{l-1}G_{l-1})P_{l-1} \cdots P_k \\ &= P_{l-1}(I - G_{l-1}P_{l-1})P_{l-2} \cdots P_k \\ &= O_{n_l \times n_k} \ ,\end{aligned}$$

whereas, if $l < k$,

$$\begin{aligned}\check{G}_l\check{P}_k(I - P_{k-1}G_{k-1}) &= G_l \cdots G_{k-1}G_k \cdots G_{J-1}P_{J-1} \cdots P_k(I - P_{k-1}G_{k-1}) \\ &= G_l \cdots G_{k-1}(I - P_{k-1}G_{k-1}) \\ &= G_l \cdots G_{k-2}(I - G_{k-1}P_{k-1})G_{k-1} \\ &= O_{n_l \times n_k} \ .\end{aligned}$$

■

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