High Resolution Difference Schemes for Compressible Gas Dynamics

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The new schemes to be presented here have grown out of an extensive comparison of a variety of difference methods. The new schemes represent an attempt to combine the advantages and avoid the disadvantages of the schemes which were compared—namely, the von-Neumann-Richtmyer scheme [1], Godunov's scheme [2], MUSCL [3,4], and Glimm's scheme [5,6,7].

We list the advantages of the various schemes first. The principal advantage of the von-Neumann-Richtmyer scheme is its use of a staggered grid. Densities, internal energies, and hence pressures also are prescribed at zone centers, while velocities are prescribed at zone interfaces. This grid structure is well suited to the Lagrangian equations of hydrodynamics, because it allows narrow-based differences to be used to construct the necessary gradients. The result is that unusually high resolution of flow structure is obtained in Lagrangian problems.

The advantage of Godunov's scheme is the clear physical picture upon which it is based. Rather than replacing an infinite Taylor series by a truncated one, this scheme replaces a physical system of complex structure by a simpler one consisting of structureless zones. This simpler system is evolved exactly for a time step, and then a similar replacement is made. Naturally, the method rests on the assumption that a Taylor series can be truncated, but the physical picture is always clear. To a physicist, this formulation has im- mense appeal. By carrying the accuracy of the physical representation one order higher than Godunov's scheme, MUSCL combines the advantage of the clear physical picture with very high resolution of the flow structure.

Glimm's scheme differs from Godunov's scheme in one essential way. For the variable values in its structureless zones it chooses those at some representative point within each zone. This point has the same location within each zone at a given time step, and it follows some well-distributed, pseudo-random sequence from one time step to another. The most important effect of this procedure is to give up exact conservation of mass, momentum, and energy in an effort to force all flow discontinuities to zone boundaries, where they can be treated exactly by the method. Because errors arising in the improper treatment of discontinuities in the flow can severely contaminate a computation with a standard difference method, the treatment of discontinuities in Glimm's method gives that scheme unequaled resolution of flow structure in one-dimensional problems.

When the above schemes are compared on a difficult two-dimensional flow problem, their disadvantages are readily apparent. The staggered grid of the von-Neumann-Richtmyer scheme, which is so convenient in Lagrangian calculations, is very badly suited to Eulerian calculations. Particularly difficult to formulate is the conservation of total energy. An additional disadvantage is the necessity to treat discontinuities as smooth flow regions with steep gradients. This is done by adding in an artificial viscous pressure which smears out the discontinuities over at least two zones. The main disadvantage of Godunov's scheme is its relatively poor resolution of flow structure. MUSCL has the highest resolution of these four schemes, but that resolution is limited by an extrapolation procedure which is made at the beginning of each time step. MUSCL uses as data a zone-centered average value and first derivative of each variable. From these, values of all variables at the zone interfaces must be constructed in order to compute fluxes of conserved quantities during the time step. The extrapolation from the center of the zone to the zone interface is responsible for most of the error in MUSCL. Finally, the disadvantage of Glimm's scheme is that its very special properties in one-dimensional problems are lost in two dimensions, and the scheme must be abandoned in favor of a much lower resolution method in the neighborhood of discontinuities (see [7]). Because the principal advantage of Glimm's scheme in 1-D flows was its treatment of discontinuities, the hybridization of the scheme for 2-D problems results in a poorer scheme than either MUSCL or the von-Neumann-Richtmyer scheme.

We have devised two new difference schemes which avoid all these disadvantages and combine the advantages listed above. The key ingredients are: (1) the approach of Godunov's method in replacing a complicated physical system with a simpler one of a standard form, (2) the translation of this assumed spatial structure inside zones into temporal structure at the interfaces by solving Riemann's problem as in Godunov's scheme and using the characteristic equations as in MUSCL, and a new ingredient (3) the use of both zone-averaged values and interface values of variables in order to define a distribution of each variable at every point which is continuous except at true flow discontinuities and which conserves mass, momentum, and energy exactly. We have devised a second-order method which uses a piecewise linear distribution for each variable with kinks at zone centers and zone interfaces. In addition we constructed a second-order method which uses a piecewise parabolic distribution for each variable with kinks only at the zone interfaces. In 1-D test problems, both new schemes show at least twice the resolution of MUSCL, the best of the four schemes discussed above. Only the piecewise parabolic scheme has been run on 2-D problems. It preserves the high resolution of its 1-D tests and is thus able to obtain a more accurate a flow description than MUSCL while using only half as many zones in each dimension. The gains over the other schemes discussed are still larger. The new algorithm is not yet optimized, but it presently requires only 30% more computer time per zone per time step than does MUSCL. The gain in time consumed to achieve a given accuracy is thus a factor of 3 in 1-D. In 2-D, a more complicated operator splitting algorithm doubles the time consumed, so that the gain is still only a factor of 3. However, we expect that the new method can be speeded up considerably, and we hope to do so in the near future.

In Fig. 1 all the schemes discussed above are compared using the example of the flow of air (gamma is 1.4) through a duct containing a step. Initially the flow is everywhere to the right at Mach 3, with $\rho = 1.4$, $p = 1$, $c = 1$. The duct width is 1, its length is 3, and the step of height .2 is located a distance of .6 from the entrance. All the results in Fig. 1 were obtained with a uniform Cartesian grid with $\Delta x = \Delta y = .05$. At the exit a "flowout" boundary condition is applied, but this is unimportant because the flow to the right is always slightly supersonic there. The system is shown at time 4, when a complicated system of shock reflections, rarefaction waves, and contact discontinuities is present. This problem was used by Emery in 1969 [8] to compare the methods of Lax, Rusanov, and Lax and Wendroff (Emery used a very slightly different duct with a grid of nodal points with $\Delta x = \Delta y = 1/27$). One of us also used this problem to demonstrate the MUSCL scheme described by van Leer in [3]. In that article the relatively structureless steady flow in the duct is displayed. This steady flow is attained at about time 12.

Because of the lack of space, we show only the contours of density at time 4. These are the most difficult to compute correctly, because of the weak contact discontinuities which emanate from the two shock triple points associated with the two Mach reflections of the bow shock at time 4. In Emery's article [8], only pressure contours are shown, and it is likely that the weak contact discontinuities in the flow were not resolved by any of the three methods he compared. In Fig. 1a the results of Godunov's scheme are shown. There is some indication of the Mach reflection at the upper wall. In Fig. 1b, the Glimm-Godunov hybrid scheme shows only some improvement over Fig. 1a at the cost of introducing noise from the random choice feature of Glimm's scheme. If the Mach reflection could be fully resolved (it is indeed resolved with 4 times as many zones), the weak contact discontinuity would be quite sharp. After an initial smearing by Godunov's method, Glimm's method preserves the relatively narrow contact region.

A dramatic increase in resolution results from using a second-order accurate scheme. The results in Fig. 1c were obtained with the BBC code [9]. This code uses a modified von-Neumann-Richtmyer scheme devised by DeBar [10] for its Lagrangian step, and a MUSCL remap step on a staggered grid devised by Woodward. To obtain the thin shocks shown here, the artifical viscosity was set to zero in the Lagrangian step. This has resulted in a mild oscillation behind each shock which is most evident when the pressure is plotted. When the artificial viscosity is turned on, the shocks double in width and the flow resolution is significantly degraded. Especially to one who considers the staggered grid formulation both confusing and inconvenient, these results are remarkably good. The contact discontinuity near the upper wall is spread over 2 to 3 zones, but it is clearly visible. Also the upper Mach stem is in the correct position directly above the step, and it has the correct length (as proven by a run on a more refined mesh of $\Delta x = \Delta y = .02$ which is shown in Fig. 2). The MUSCL results shown in Fig. 1d are of comparable quality. They are superior in that the post-shock oscillations of BBC are not present and the contact discontinuity is more sharply defined. However, somewhat more entropy is artificially produced as the flow rounds the corner of the step. The result is a classic interaction of a shock with a boundary layer, which produces the second, very weak reflected shock from the top of the step at $x = 1.4$. These two codes run at almost precisely the same speed—2800 points per sec per cycle on a CDC 7600 and 20000 pts/sec/cy on a Cray I. They both make use of separate Lagrangian and remap steps in each 1-D pass.

In Fig. 1e we show the results of the new scheme which uses piecewise parabolic interpolation. In 1-D this scheme can be made third-order accurate, but because of its use of 1-D passes it can only be second-order accurate in 2-D. The scheme used here is thus made only second-order accurate in 1-D, although several gestures toward higher order are included. The results of this new scheme are comparable in quality to those of Fig. 2, which were obtained with BBC using a much finer grid of $\Delta x = \Delta y = .02$. The resolution of the weak contact discontinuities from both Mach stems is particularly notable. A "monotonicity trick" has been used to constrain the interpolation parabolae so that the post-shock oscillations usually associated with high-order schemes are completely absent. In Fig. 3 we show results of the new scheme using a grid with $\Delta x = \Delta y = .1$. Evidently, even on this coarse grid the new scheme correctly resolves all the essential features of this complicated flow.

Finally, for comparison with the schemes discussed by Sod [11], we have shown in Fig. 4 results of the new piecewise linear scheme on his shock tube problem. The results in Fig. 4 use a grid of 50 zones rather than Sod's 100, and are more accurate than the results of any of the 12 schemes he compared. MUSCL results on this problem have been given by van Leer [3].

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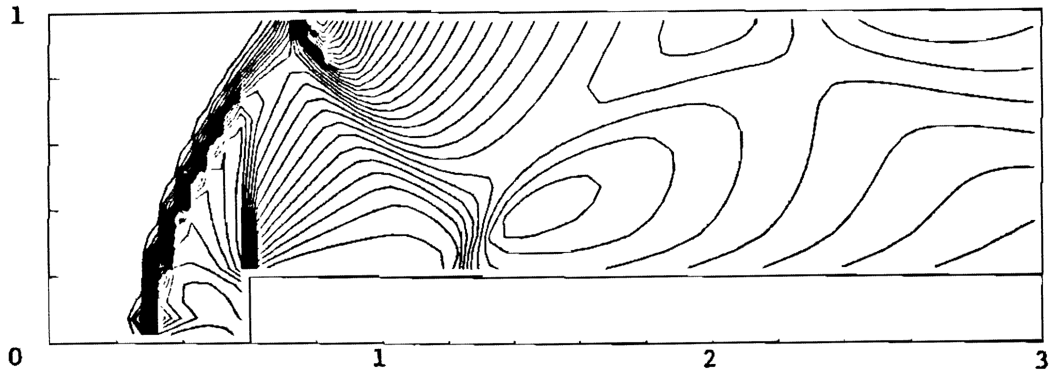
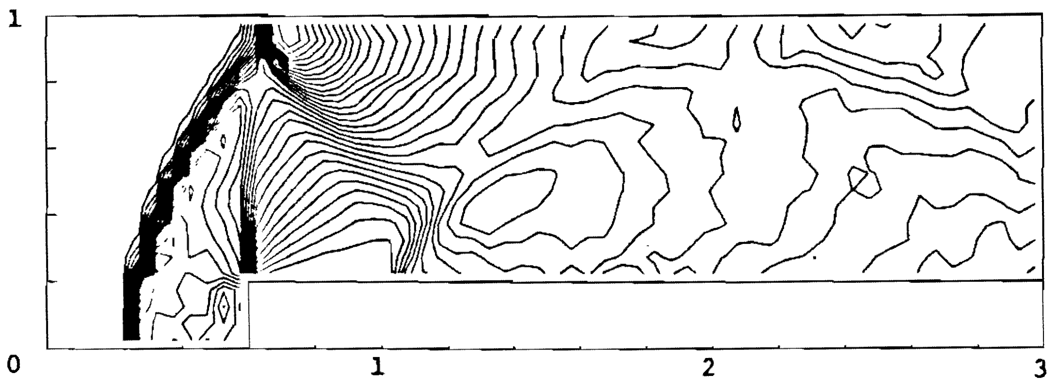
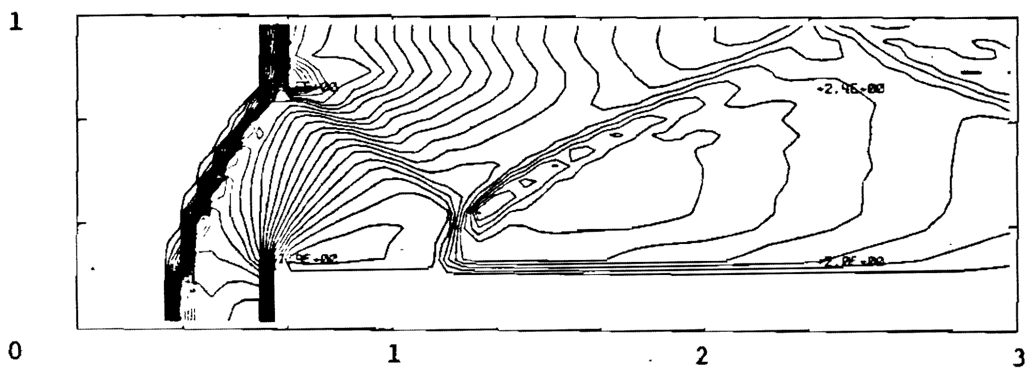


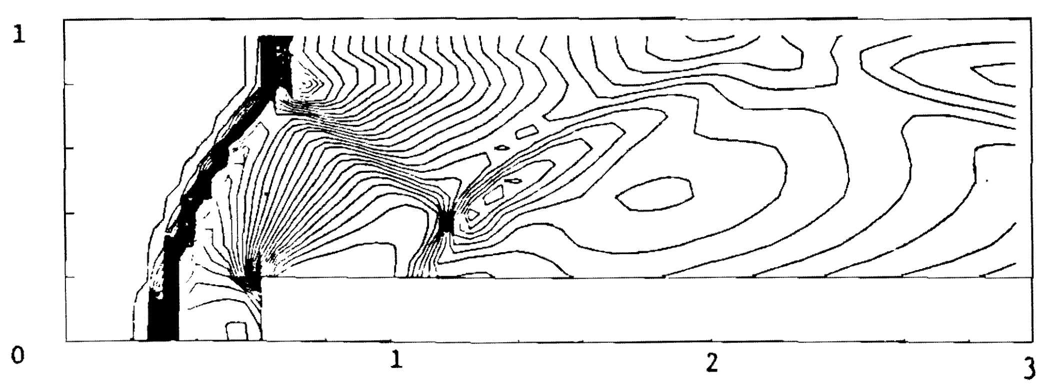
1a) Godunov's Method 29 density contours from 1.18 to 7.14



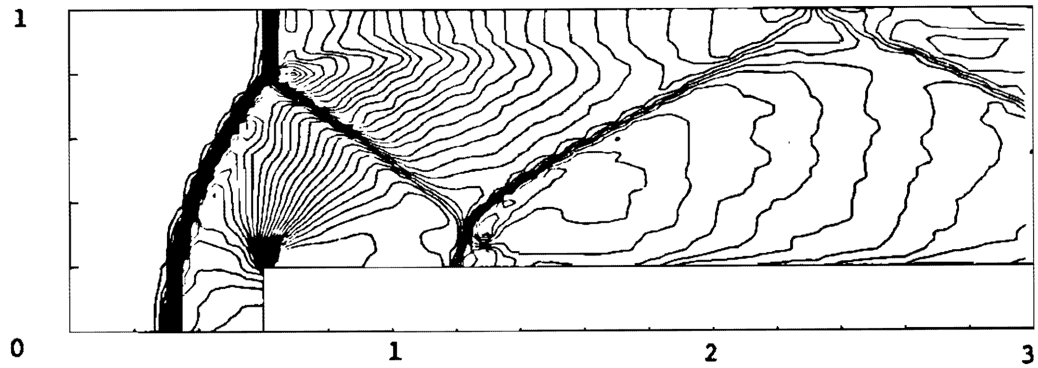
1b) Glimm-Godunov Hybrid 29 density contours from 1.15 to 6.74



1c) BBC 22 density contours from 1.00 to 6.25



1d) MUSCL 30 density contours from 0.85 to 6.34



1e) Piecewise Parabolic Method 30 density contours from 0.73 to 6.31

Fig. 1. Results of several difference schemes for the flow problem described in the text. All schemes use a series of 1-D sweeps and Courant numbers of 0.8 or 0.9. All use a uniform grid of 20 x 60 zones. The methods are: (a) Godunov's method, (b) Glimm-Godunov hybrid, (c) BBC, (d) MUSCL, (e) the new piecewise parabolic scheme.

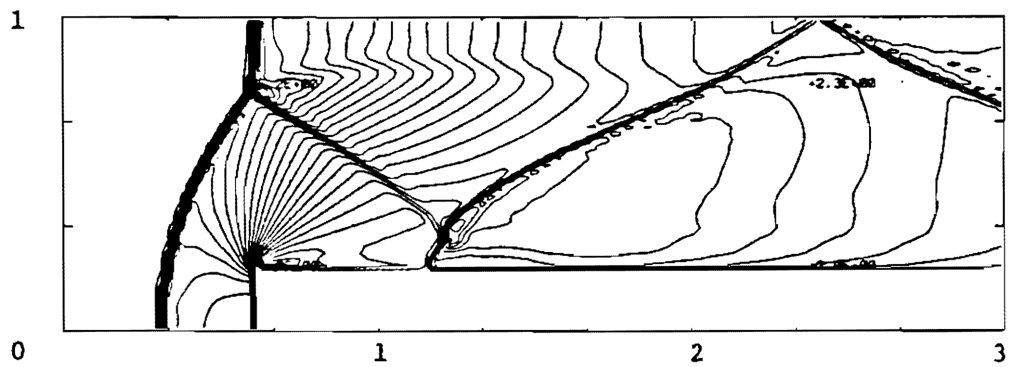


Fig. 2 BBC results for the same problem as in Fig. 1 but using a finer, uniform grid of 50 x 150 zones.

23 density contours from 0.75 to 6.25 are shown as well as contours at densities 0.5 and 0.6.

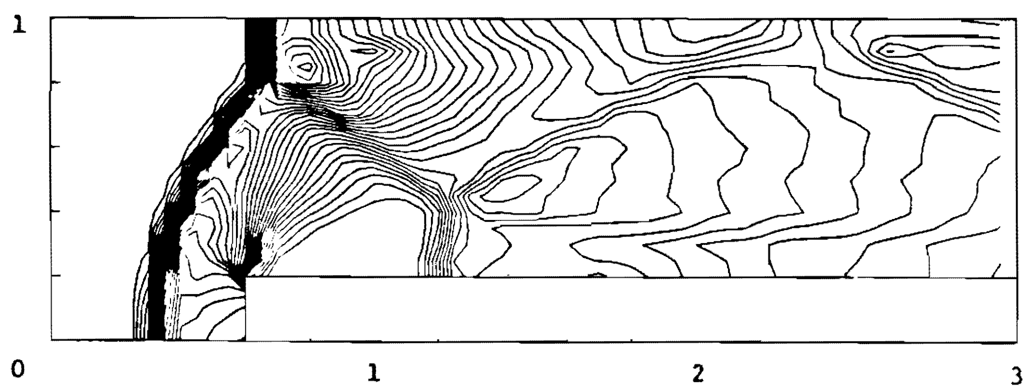


Fig. 3 Results of the new piecewise parabolic scheme for the same problem as in Fig. 1 but using a coarser, uniform grid of 10 x 30 zones.

30 density contours from 1.03 to 6.03 are shown.

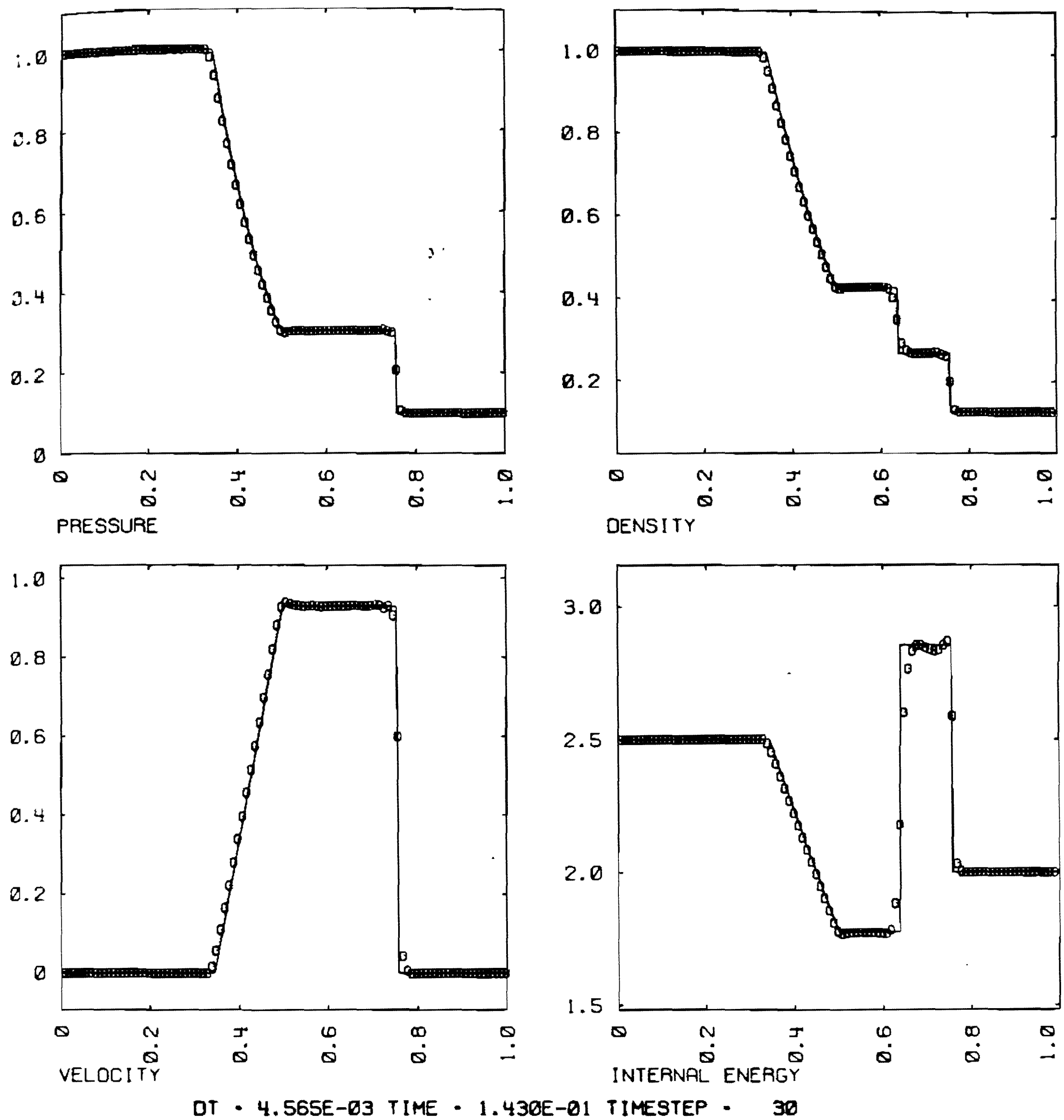


Fig. 4 Results of the new piecewise linear scheme for the shock tube problem studied by Sod. The solid line shows the exact solution, and the zone average and interface values for the grid of 50 zones are shown as circles.