POLYLOGARITHMS
AND
ASSOCIATED FUNCTIONS

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APPENDIX

REFERENCE DATA AND TABLES

A.1 GLOSSARY OF NOTATION

The following symbols have been used throughout the book and are collected here for ease of reference. Whenever possible, standard usage has been adhered to.

(1) B_n. The nth Bernoulli number, defined in terms of the coefficients of the power series expansion of

$$\frac{z}{e^z-1}=1-\tfrac{1}{2}z+\sum_1^\infty(-1)^{n-1}B_n z^{2n}/(2n)!$$

(2) $B_n(z)$. The nth Bernoulli polynomial, defined as the coefficient of $t^n/n!$ in the power series expansion of $te^{zt}/(e^z-1)$

(3) $\mathrm{Cl}_n(\theta)$. The generalized Clausen function. If n is even,

$$\mathrm{Cl}_n(\theta)=\sum_1^\infty\frac{\sin r\theta}{r^n};\ \text{if } n \text{ is odd, } \mathrm{Cl}_n(\theta)=\sum_1^\infty\frac{\cos r\theta}{r^n}$$

(4) $D_n(x,\alpha)=\int_o^x\frac{\log^{n-1}|x|(\cos\alpha+x)\,dx}{1+2x\cos\alpha+x^2}$, Kummer's function related to $\mathrm{Re}\,\Lambda_n(xe^{i\alpha})$

(5) $E_n(x,\alpha)=\int_o^x\frac{\log^{n-1}|x|\sin\alpha\,dx}{1+2x\cos\alpha+x^2}$, Kummer's function related to $\mathrm{Im}\,\Lambda_n(xe^{i\alpha})$

(6) E_n, the nth Euler number, defined in terms of the coefficients of the power series expansion of $\sec z=1+\sum_1^\infty E_r z^{2r}/(2r)!$

(7) G. Catalan's constant, $=\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\cdots$

(8) $\mathrm{Gl}_n(\theta)$. The associated Clausen function. If n is even, $\mathrm{Gl}_n(\theta)=\sum_1^\infty \frac{\cos r\theta}{r^n}$; if n is odd, $\mathrm{Gl}_n(\theta)=\sum_1^\infty \frac{\sin r\theta}{r^n}$

(9) γ. Euler's constant, $=\lim_{n\to\infty}\left[\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)-\log n\right]$

(10) $i=\sqrt{-1}$

(11) Im = imaginary part of

(12) $L(x)$. Newman's function, $=\int_1^x -\frac{\log(x)}{1-x}\,dx=-\mathrm{Li}_2(1-x)$ (*Ad hoc* uses of L also occur.)

(13) $\overset{n}{L}(1+x)$. Spence's function, $=\frac{x}{1^n}-\frac{x^2}{2^n}+\frac{x^3}{3^n}-\cdots=-\mathrm{Li}_n(-x)$

(14) $\log(x)$. The Napierian logarithm (only logarithms to base e are used)

(15) $\log^n(x)\equiv[\log(x)]^n$.

(16) $\mathrm{Li}_n(x)$. The polylogarithm of order n, $=\frac{x}{1^n}+\frac{x^2}{2^n}+\frac{x^3}{3^n}+\cdots=\int_o^x \mathrm{Li}_{n-1}(x)\,dx/x$

(17) $\mathrm{Li}_n(r,\theta)=\mathrm{Re}\,\mathrm{Li}_n(re^{i\theta})$

(18) $\mathrm{Ls}_n(\theta)$. The log-sine integral of order n, $=-\int_o^\theta \log^{n-1}|2\sin\frac{1}{2}\theta|\,d\theta$

(19) $\mathrm{Ls}_3(\theta,\alpha)$. The extended log-sine integral of the third order of argument θ and parameter α, $=-\int_o^\theta \log|2\sin\frac{1}{2}\theta|\log|2\sin(\frac{1}{2}\theta+\frac{1}{2}\alpha)|\,d\theta$

(20) $\mathrm{Ls}_n^{(m)}(\theta)$. The generalized log-sine integral, of order n and index m, $=-\int_o^\theta \theta^m\log^{n-m-1}|2\sin\frac{1}{2}\theta|\,d\theta$

(21) $\Lambda_n(x)$. Kummer's function, $=\int_o^x \frac{\log^{n-1}|x|}{1+x}\,dx$

(22) $[x]$ = the integral part of x

(23) $\Psi(x)$. The logarithmic derivative of the Gamma-function, $=\Gamma'(x)/\Gamma(x)$

(24) Re = real part of

(25) $\mathrm{sgn}(y)\equiv+$ if y is positive, $-$ if y is negative

(26) $\mathrm{Ti}_n(y)$. The inverse tangent integral of order n, $=\frac{y}{1^n}-\frac{y^3}{3^n}+\frac{y^5}{5^n}-\cdots$ $=\int_o^y \mathrm{Ti}_{n-1}(y)\,dy/y$; $\mathrm{Ti}_2(y)=\int_o^y \tan^{-1}(y)\,dy/y$

(27) $\mathrm{Ti}_2(y,a)$. The generalized inverse tangent integral of order 2, $=\int_o^y \frac{\tan^{-1}y}{y+a}\,dy$

(28) $\mathrm{Ti}_n(y,a)$. The generalized inverse tangent integral of order n, $=\int_o^y\left[\frac{\mathrm{Ti}_{n-1}(y)}{y+a}+\frac{\mathrm{Ti}_{n-1}(y,a)}{y}-\frac{\mathrm{Ti}_{n-1}(y,a)}{y+a}\right]dy$

(29) $\chi_n(x)$. Legendre's chi-function of order n, $=\frac{x}{1^n}+\frac{x^3}{3^n}+\frac{x^5}{5^n}+\cdots$

(30) $\zeta(s)$. Riemann's zeta-function, $=\sum_1^\infty n^{-s}$

(31) $\zeta(s,a)$. The generalized Riemann zeta-function, $=\sum_0^\infty (n+a)^{-s}$

A.2. LIST OF SELECTED FORMULAS

The following list of formulas is selected from the results given earlier in the book, or from the literature in the case of a few equations whose derivation is so close to that of those already given that further analysis should be unnecessary. In order to keep the list reasonably short, references are made when appropriate to the relevant sections for further equations similar to those quoted. In general the results are given approximately in the order in which they occur, chapter by chapter.

A.2.1. The Dilogarithm

(1) $\mathrm{Li}_2(x)=-\int_o^x \frac{\log(1-z)}{z}\,dz=\frac{x}{1^2}+\frac{x^2}{2^2}+\frac{x^3}{3^2}+\cdots$ when $|x|\leqq 1$

(2) $\mathrm{Li}_2(1)=\pi^2/6$

(3) $\mathrm{Li}_2(-1)=-\pi^2/12$

(4) $\mathrm{Li}_2(\frac{1}{2})=\pi^2/12-\frac{1}{2}\log^2(2)$

Further numerical relations are given in Section 1.4.

(5) $\mathrm{Li}_2(-x)+\mathrm{Li}_2(-1/x)=-\pi^2/6-\frac{1}{2}\log^2(x)$, $\quad x>0$

(6) $\mathrm{Li}_2(x)+\mathrm{Li}_2(1/x)=\pi^2/3-\frac{1}{2}\log^2(x)-i\pi\log x$, $\quad x>1$

(7) $\mathrm{Li}_2(x)+\mathrm{Li}_2(1-x)=\pi^2/6-\log x\log(1-x)$

(8) $\mathrm{Li}_2(x)+\mathrm{Li}_2\left(-\frac{x}{1-x}\right)=-\frac{1}{2}\log^2(1-x)$, $\quad x<1$

(9) $\mathrm{Li}_2(x)+\mathrm{Li}_2\left(\frac{x}{x-1}\right)=\frac{1}{2}\pi^2-\frac{1}{2}\log^2(x-1)+i\pi\log\left(\frac{x-1}{x^2}\right)$, $\quad x>1$

(10) $\mathrm{Li}_2(-x)-\mathrm{Li}_2(1-x)+\frac{1}{2}\mathrm{Li}_2(1/x^2)=\log x\log\left(\frac{x-1}{x}\right)$, $\quad x>1$

(11) $\mathrm{Li}_2(1-x)-\mathrm{Li}_2(1/x)=\frac{1}{2}\log x\log\left[\frac{x}{(x-1)^2}\right]-\pi^2/6$, $\quad x>1$

(12) $\mathrm{Li}_2\left(\frac{1}{1+x}\right)-\mathrm{Li}_2(-x)=\pi^2/6-\frac{1}{2}\log(1+x)\log\left(\frac{1+x}{x^2}\right)$, $\quad x>0$

(13) $\mathrm{Li}_2(x)+\mathrm{Li}_2(-x)=\frac{1}{2}\mathrm{Li}_2(x^2)$

(14) $\mathrm{Li}_2\left(\frac{x}{1+x}\right)+\mathrm{Li}_2\left(\frac{-x}{1-x}\right)=\frac{1}{2}\mathrm{Li}_2\left(\frac{-x^2}{1-x^2}\right)-\frac{1}{4}\log^2\left(\frac{1+x}{1-x}\right)$, $-1<x<1$

(15) $\mathrm{Li}_2(2x-x^2)+2\,\mathrm{Li}_2\left(\frac{1}{2-x}\right)-2\,\mathrm{Li}_2(x)=\pi^2/6-\log^2(2-x), \qquad x<1$

Further relations of this nature are given in Section 1.5.

(16) $\mathrm{Li}_2\left(\frac{x}{1-x}\cdot\frac{y}{1-y}\right)=\mathrm{Li}_2\left(\frac{x}{1-y}\right)+\mathrm{Li}_2\left(\frac{y}{1-x}\right)-\mathrm{Li}_2(x)-\mathrm{Li}_2(y)$
$-\log(1-x)\log(1-y)$ (Abel)

(17) $\mathrm{Li}_2(xy)=\mathrm{Li}_2(x)+\mathrm{Li}_2(y)-\mathrm{Li}_2\left[\frac{x(1-y)}{1-xy}\right]-\mathrm{Li}_2\left[\frac{y(1-x)}{1-xy}\right]$
$-\log\left(\frac{1-x}{1-xy}\right)\log\left(\frac{1-y}{1-xy}\right)$ (Hill)

(18) $\mathrm{Li}_2\left[\frac{y(1-x)}{x(1-y)}\right]=\mathrm{Li}_2(x)-\mathrm{Li}_2(y)+\mathrm{Li}_2(y/x)+\mathrm{Li}_2\left(\frac{1-x}{1-y}\right)-\frac{\pi^2}{6}$
$+\log x\log\left(\frac{1-x}{1-y}\right)$ (Schaeffer)

(19) $\mathrm{Li}_2\left[\frac{x(1-y)^2}{y(1-x)^2}\right]=\mathrm{Li}_2\left[-\frac{x(1-y)}{1-x}\right]+\mathrm{Li}_2\left[-\frac{(1-y)}{y(1-x)}\right]$
$+\mathrm{Li}_2\left[\frac{x}{y}\frac{(1-y)}{(1-x)}\right]+\mathrm{Li}_2\left(\frac{1-y}{1-x}\right)+\frac{1}{2}\log^2(y)$ (Kummer)

Other relations of a similar nature will be found in Sections 1.5 and 8.3.1, where ranges of validity are discussed.

(20) $2\,\mathrm{Li}_2(x)+2\,\mathrm{Li}_2(y)+2\,\mathrm{Li}_2(z)=\mathrm{Li}_2(-xy/z)+\mathrm{Li}_2(-yz/x)+\mathrm{Li}_2(-zx/y)$, where $1/x+1/y+1/z=1$ (Newman)

(21) $2\,\mathrm{Li}_2(xy)+2\,\mathrm{Li}_2(yz)+2\,\mathrm{Li}_2(zx)=\mathrm{Li}_2(-x^2)+\mathrm{Li}_2(-y^2)+\mathrm{Li}_2(-z^2)$, where $x+y+z-xyz=0$

(22) $\mathrm{Li}_2(y)=\pi^2/6+\sum_1^N\sum_1^N[\mathrm{Li}_2(x_m\lambda_n)-\mathrm{Li}_2(\lambda_m/\lambda_n)]$, where the x_m are the roots of the equation $\prod_1^N(1-\lambda_n x)=1-y$

(23) $\chi_2(x)=\frac{1}{2}\int_o^x\log\left(\frac{1+x}{1-x}\right)\frac{dx}{x}=\frac{1}{2}\mathrm{Li}_2(x)-\frac{1}{2}\mathrm{Li}_2(-x)$
$=\frac{x}{1^2}+\frac{x^3}{3^2}+\frac{x^5}{5^2}+\cdots, |x|\leqq 1$

(24) $\chi_2(x)+\chi_2\left(\frac{1-x}{1+x}\right)=\pi^2/8+\frac{1}{2}\log\left(\frac{1+x}{1-x}\right)\log x$

(25) $\chi_2(1)=\pi^2/8$

Other numerical relations for $\chi_2(x)$ are given in Section 1.8.

(26) $\chi_2(e^{-2i\theta})+\chi_2(i\tan\theta)=\pi^2/8+i\theta\log(i\tan\theta)$

(27) $\mathrm{Li}_2(e^{-z})=\pi^2/6+(z\log z-z)-\frac{z^2}{4}+\frac{B_1z^3}{2.3.2!}-\frac{B_2z^5}{4.5.4!}+\cdots,$
$|z|<2\pi$

A.2.2. The Inverse Tangent Integral of Second Order

(1) $\mathrm{Ti}_2(y)=\int_o^y\frac{\tan^{-1}y}{y}\,dy=\frac{y}{1^2}-\frac{y^3}{3^2}+\frac{y^5}{5^2}-\cdots$ when $|y|\leqq 1$

(2) $\mathrm{Ti}_2(1)=G=0.9159655942$

(3) $\mathrm{Ti}_2(\tan\pi/12)=\mathrm{Ti}_2(2-\sqrt{3})=2G/3+(\pi/12)\log\tan(\pi/12)$

Other results of a similar nature will be found in Sections 2.5 and 2.6.

(4) $\mathrm{Ti}_2(y)-\mathrm{Ti}_2(1/y)=\mathrm{sgn}(y)\frac{\pi}{2}\log|y|, \qquad y$ real

(5) $\mathrm{Ti}_2(\frac{1}{2}y^2)+\frac{1}{2}\mathrm{Ti}_2\left[\frac{y(2+y)}{2(1+y)}\right]-\frac{1}{2}\mathrm{Ti}_2\left[\frac{y(2-y)}{2(1-y)}\right]+\mathrm{Ti}_2\left(\frac{y}{2+y}\right)$
$-\mathrm{Ti}_2\left(\frac{y}{2-y}\right)+\mathrm{Ti}_2(1-y)+\mathrm{Ti}_2\left(\frac{1}{1+y}\right)=2G+\frac{\pi}{4}\log\left(\frac{1-y}{1+y}\right),$
$-1<y<1$ (Spence)

An extension of the range in y is given in Section 2.3.

(6) $2\,\mathrm{Ti}_2(\frac{1}{2})+\mathrm{Ti}_2(\frac{1}{3})+\frac{1}{2}\mathrm{Ti}_2(\frac{3}{4})=3G-\frac{1}{2}\pi\log 2$

(7) $\frac{1}{3}\mathrm{Ti}_2(\tan 3\theta)=\mathrm{Ti}_2(\tan\theta)+\mathrm{Ti}_2\left[\tan\left(\frac{\pi}{6}-\theta\right)\right]-\mathrm{Ti}_2\left[\tan\left(\frac{\pi}{6}+\theta\right)\right]$
$+\frac{\pi}{6}\log\left\{\frac{\tan[(\pi/6)+\theta]}{\tan[(\pi/6)-\theta]}\right\}, \qquad -\pi/6<\theta<\pi/6$

(8) $\frac{1}{2n+1}\mathrm{Ti}_2[\tan(\overline{2n+1}\,\theta)]=\mathrm{Ti}_2(\tan\theta)$
$-\sum_1^n\mathrm{Ti}_2\left[\tan\left(\frac{\overline{2r-1}\pi}{2(2n+1)}+\theta\right)\right]+\sum_1^n\mathrm{Ti}_2\left[\tan\left(\frac{\overline{2r-1}\pi}{2(2n+1)}-\theta\right)\right]$
$+\frac{\frac{1}{2}\pi}{2n+1}\sum_1^n(2r-1)\log\frac{\tan\left[\frac{\overline{2r-1}\pi}{2(2n+1)}+\theta\right]}{\tan\left[\frac{\overline{2r-1}\pi}{2(2n+1)}-\theta\right]}, \quad -\frac{\frac{1}{2}\pi}{2n+1}<\theta<\frac{\frac{1}{2}\pi}{2n+1}$

(15) $\mathrm{Li}_2(2x-x^2)+2\,\mathrm{Li}_2\left(\frac{1}{2-x}\right)-2\,\mathrm{Li}_2(x)=\pi^2/6-\log^2(2-x), \qquad x<1$

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$+\mathrm{Li}_2\left[\frac{x}{y}\frac{(1-y)}{(1-x)}\right]+\mathrm{Li}_2\left(\frac{1-y}{1-x}\right)+\frac{1}{2}\log^2(y)$ (Kummer)

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$=\frac{x}{1^2}+\frac{x^3}{3^2}+\frac{x^5}{5^2}+\cdots, |x|\leqq 1$

(24) $\chi_2(x)+\chi_2\left(\frac{1-x}{1+x}\right)=\pi^2/8+\frac{1}{2}\log\left(\frac{1+x}{1-x}\right)\log x$

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(26) $\chi_2(e^{-2i\theta})+\chi_2(i\tan\theta)=\pi^2/8+i\theta\log(i\tan\theta)$

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A.2.2. The Inverse Tangent Integral of Second Order

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(2) $\mathrm{Ti}_2(1)=G=0.9159655942$

(3) $\mathrm{Ti}_2(\tan\pi/12)=\mathrm{Ti}_2(2-\sqrt{3})=2G/3+(\pi/12)\log\tan(\pi/12)$

Other results of a similar nature will be found in Sections 2.5 and 2.6.

(4) $\mathrm{Ti}_2(y)-\mathrm{Ti}_2(1/y)=\mathrm{sgn}(y)\frac{\pi}{2}\log|y|, \qquad y$ real

(5) $\mathrm{Ti}_2(\frac{1}{2}y^2)+\frac{1}{2}\mathrm{Ti}_2\left[\frac{y(2+y)}{2(1+y)}\right]-\frac{1}{2}\mathrm{Ti}_2\left[\frac{y(2-y)}{2(1-y)}\right]+\mathrm{Ti}_2\left(\frac{y}{2+y}\right)$
$-\mathrm{Ti}_2\left(\frac{y}{2-y}\right)+\mathrm{Ti}_2(1-y)+\mathrm{Ti}_2\left(\frac{1}{1+y}\right)=2G+\frac{\pi}{4}\log\left(\frac{1-y}{1+y}\right),$
$-1<y<1$ (Spence)

An extension of the range in y is given in Section 2.3.

(6) $2\,\mathrm{Ti}_2(\frac{1}{2})+\mathrm{Ti}_2(\frac{1}{3})+\frac{1}{2}\mathrm{Ti}_2(\frac{3}{4})=3G-\frac{1}{2}\pi\log 2$

(7) $\frac{1}{3}\mathrm{Ti}_2(\tan 3\theta)=\mathrm{Ti}_2(\tan\theta)+\mathrm{Ti}_2\left[\tan\left(\frac{\pi}{6}-\theta\right)\right]-\mathrm{Ti}_2\left[\tan\left(\frac{\pi}{6}+\theta\right)\right]$
$+\frac{\pi}{6}\log\left\{\frac{\tan[(\pi/6)+\theta]}{\tan[(\pi/6)-\theta]}\right\}, \qquad -\pi/6<\theta<\pi/6$

(8) $\frac{1}{2n+1}\mathrm{Ti}_2[\tan(\overline{2n+1}\,\theta)]=\mathrm{Ti}_2(\tan\theta)$
$-\sum_1^n\mathrm{Ti}_2\left[\tan\left(\frac{\overline{2r-1}\pi}{2(2n+1)}+\theta\right)\right]+\sum_1^n\mathrm{Ti}_2\left[\tan\left(\frac{\overline{2r-1}\pi}{2(2n+1)}-\theta\right)\right]$
$+\frac{\frac{1}{2}\pi}{2n+1}\sum_1^n(2r-1)\log\frac{\tan\left[\frac{\overline{2r-1}\pi}{2(2n+1)}+\theta\right]}{\tan\left[\frac{\overline{2r-1}\pi}{2(2n+1)}-\theta\right]}, \qquad -\frac{\frac{1}{2}\pi}{2n+1}<\theta<\frac{\frac{1}{2}\pi}{2n+1}$

$$(9)\ \mathrm{Ti}_2\left[\frac{4y(1+y)(2+y)}{(2-y^2)(2+4y+y^2)}\right]+\mathrm{Ti}_2\left[\frac{4y(1-y)(2-y)}{(2-y^2)(2-4y+y^2)}\right]$$
$$+2\,\mathrm{Ti}_2\left(\frac{2+4y+y^2}{2-y^2}\right)-2\,\mathrm{Ti}_2\left(\frac{2-4y+y^2}{2-y^2}\right)+4\,\mathrm{Ti}_2\left(\frac{y}{2+y}\right)$$
$$+4\,\mathrm{Ti}_2\left(\frac{y}{2-y}\right)-4\,\mathrm{Ti}_2(1+y)+4\,\mathrm{Ti}_2(1-y)-8\,\mathrm{Ti}_2\left(\frac{2y}{2-y^2}\right)$$
$$=\tfrac{1}{2}\pi\log\left(\frac{2+4y+y^2}{2-4y+y^2}\right)+\pi\log\left(\frac{1-y}{1+y}\right),\qquad -(2-\sqrt{2})<y<(2-\sqrt{2})$$

$$(10)\ \mathrm{Ti}_2\left[\frac{4y(1+y)(2+y)}{(2-y^2)(2+4y+y^2)}\right]+2\,\mathrm{Ti}_2\left(\frac{2+4y+y^2}{2-y^2}\right)+4\,\mathrm{Ti}_2\left(\frac{y}{2+y}\right)$$
$$-4\,\mathrm{Ti}_2(1+y)-4\,\mathrm{Ti}_2\left(\frac{2y}{2-y^2}\right)+2\,\mathrm{Ti}_2\left(\frac{2-y^2}{2+y^2}\right)+\mathrm{Ti}_2\left(\frac{4y^2}{4-y^4}\right)$$
$$=\tfrac{1}{2}\pi\log\left[\frac{2+4y+y^2}{(2+y^2)(1+y)^2}\right],\qquad -2+\sqrt{2}<y<\sqrt{2}$$

An extension of the range in y is given in Section 2.7.

$$(11)\ \mathrm{Ti}_2(7/24)+2\,\mathrm{Ti}_2(1/7)+6\,\mathrm{Ti}_2(1/3)-8\,\mathrm{Ti}_2(1/2)+\mathrm{Ti}_2(3/4)$$
$$+\tfrac{1}{2}\pi\log(3/2)=0$$

$$(12)\ 2\,\mathrm{Ti}_2\left(\frac{2\sqrt{2}y}{1-y^2}\right)=\mathrm{Ti}_2\left[\frac{\alpha^2+y}{\alpha(1-y)}\right]-\mathrm{Ti}_2\left[\frac{\alpha^2-y}{\alpha(1+y)}\right]+\mathrm{Ti}_2\left[\frac{\alpha(1-y)}{1+\alpha^2y}\right]$$
$$-\mathrm{Ti}_2\left[\frac{\alpha(1+y)}{1-\alpha^2y}\right]+\tfrac{1}{2}\mathrm{Ti}_2\left(\frac{2y\alpha}{1-y^2\alpha^2}\right)+\tfrac{1}{2}\mathrm{Ti}_2\left(\frac{2y\alpha}{\alpha^2-y^2}\right)$$
$$+\tfrac{1}{4}\mathrm{Ti}_2\left[\frac{4y\alpha(1-\alpha^2y^2)}{(1-y^2)(1-\alpha^4y^2)}\right]+\tfrac{1}{4}\mathrm{Ti}_2\left[\frac{4y\alpha(\alpha^2-y^2)}{(1-y^2)(\alpha^4-y^2)}\right]$$
$$-\frac{\pi}{8}\log\left[\frac{(\alpha^2+y)(1-\alpha^2y)}{(\alpha^2-y)(1+\alpha^2y)}\right],\qquad -\alpha^2<y<\alpha^2,\quad \alpha=\tan(\pi/8)=\sqrt{2}-1$$

(13) $\cos^2(\tfrac{1}{2}M\pi)\,\mathrm{Ti}_2(y)=\sum_{n=1}^{M+1}\left[\sum_{m=1}^{M}\mathrm{Ti}_2(\alpha_m x_n)-\mathrm{Ti}_2(x_n)\right]$, where the x_n are the roots of $\dfrac{x+y}{1-xy}=\dfrac{S_1x-S_3x^3+\cdots}{1-S_2x^2+S_4x^4-\cdots}$ and S_r is the sum of the products of the a_m taken r at a time

$$(14)\ \sum_1^3\left[\mathrm{Ti}_2\left(\frac{A}{x_nx_{n-1}}\right)+\mathrm{Ti}_2\left(\frac{x_ny}{A}\right)\right]=\mathrm{Ti}_2(y)+\sum_1^3\mathrm{Ti}_2(x_n),$$

where $x_0\equiv x_3$, $A=\tfrac{1}{2}(S+\sqrt{S^2-4P})$, $S=x_1+x_2+x_3+y$, $P=x_1x_2x_3y$, and $\tan^{-1}(x_1)+\tan^{-1}(x_2)+\tan^{-1}(x_3)+\tan^{-1}(y)=0$

The range of validity of the above two formulae is discussed in Section 2.8.

$$(15)\ \mathrm{Ti}_2\left[-\frac{1-z+\frac{1}{2}wz}{1-w+\frac{1}{2}wz}\right]+\mathrm{Ti}_2\left[-\frac{1-w+\frac{1}{2}wz}{1-z+\frac{1}{2}wz}\right]+2\,\mathrm{Ti}_2(\tfrac{1}{2}wz)+\mathrm{Ti}_2(1-w)$$
$$+\mathrm{Ti}_2(1-z)-\mathrm{Ti}_2\left(\frac{w}{2-w}\right)-\mathrm{Ti}_2\left(\frac{z}{2-z}\right)-\tfrac{1}{2}\mathrm{Ti}_2\left[\frac{2(1-w)}{w(2-w)}\right]$$
$$-\tfrac{1}{2}\mathrm{Ti}_2\left[\frac{2(1-z)}{z(2-z)}\right]+\mathrm{Ti}_2\left[\frac{1-(1-w)\frac{1}{2}wz}{1-w+\frac{1}{2}wz}\right]+\mathrm{Ti}_2\left[\frac{1-(1-z)\frac{1}{2}wz}{1-z+\frac{1}{2}wz}\right]$$
$$-\mathrm{Ti}_2\left[-\frac{w(1-z+\frac{1}{2}wz)}{2-w+\frac{1}{2}w^2z}\right]-\mathrm{Ti}_2\left[-\frac{z(1-w+\frac{1}{2}wz)}{2-z+\frac{1}{2}wz^2}\right]$$
$$-\tfrac{1}{2}\mathrm{Ti}_2\left[-\frac{2(1-\overline{1-w}\frac{1}{2}wz)(1-w+\frac{1}{2}wz)}{w(2-w+\frac{1}{2}w^2z)(1-z+\frac{1}{2}wz)}\right]$$
$$-\tfrac{1}{2}\mathrm{Ti}_2\left[-\frac{2(1-\overline{1-z}\frac{1}{2}wz)(1-z+\frac{1}{2}wz)}{z(2-z+\frac{1}{2}wz^2)(1-w+\frac{1}{2}wz)}\right]$$
$$=2\,\mathrm{Ti}_2(1)+\pi\,\mathrm{sgn}(w-z)\log\left[-\frac{z(1-w+\frac{1}{2}zw)}{w(1-z+\frac{1}{2}zw)}\right]$$
$$-\frac{\pi}{4}\log\left[\frac{w^2z^2(1-z+\frac{1}{2}wz)(1-w+\frac{1}{2}wz)}{(2-w)(2-z)(2-w+\frac{1}{2}w^2z)(2-z+\frac{1}{2}wz^2)}\right]$$

The range of validity of this formula is discussed in Section 2.8.5.

A.2.3. The Generalized Inverse Tangent Integral of Second Order

(1) $\mathrm{Ti}_2(x,a)=\int_0^x\dfrac{\tan^{-1}(x)}{x+a}\,dx$

(2) $\mathrm{Ti}_2(x,0)=\mathrm{Ti}_2(x)$

(3) $\mathrm{Ti}_2(x,\infty)=0$

(4) $\mathrm{Ti}_2(-x,a)=-\mathrm{Ti}_2(x,-a)$

(5) $\dfrac{\partial}{\partial a}\mathrm{Ti}_2(x,a)=\dfrac{\tan^{-1}(x)}{x+a}-\dfrac{1}{1+a^2}\left[\log\left(\dfrac{a+x}{a}\right)\right.$
$\left.-\tfrac{1}{2}\log(1+x^2)+a\tan^{-1}(x)\right]$

(6) $\mathrm{Ti}_2(x,a)-\mathrm{Ti}_2(a,x)=\mathrm{Ti}_2(x)-\mathrm{Ti}_2(a)+\tan^{-1}(a)\log\left[\dfrac{a(1+x^2)^{1/2}}{a+x}\right]$
$-\tan^{-1}(x)\log\left[\dfrac{x(1+a^2)^{1/2}}{x+a}\right]$

(7) $\mathrm{Ti}_2(1,a)=\frac{1}{2}\mathrm{Ti}_2\left(\frac{1-a}{1+a}\right)-\frac{1}{2}\mathrm{Ti}_2(a)-\frac{1}{4}\mathrm{Ti}_2\left(\frac{2a}{1-a^2}\right)$
$+\frac{1}{2}G+\tan^{-1}(a)\log\left(\frac{a\sqrt{2}}{1+a}\right)-\frac{\pi}{8}\log\left[\frac{1-a^4}{(1+a)^4}\right]$

(8) $\mathrm{Ti}_2(1/a,a)=\frac{1}{2}\tan^{-1}(1/a)\log\left(\frac{1+a^2}{a^2}\right)$

(9) $\mathrm{Ti}_2\left(\frac{1-a}{1+a},a\right)=\frac{1}{2}\mathrm{Ti}_2\left(\frac{1-a}{1+a}\right)-\frac{1}{2}\mathrm{Ti}_2(a)-\frac{1}{4}\mathrm{Ti}_2\left(\frac{2a}{1-a^2}\right)+\frac{1}{2}G$
$+\frac{1}{2}\tan^{-1}(a)\log\left(\frac{2a^2}{1+a^2}\right)+\frac{\pi}{8}\log\left(\frac{1+a^2}{1-a^2}\right)$

(10) $\lim_{\varepsilon\to 0}\mathrm{Ti}_2(a-\varepsilon,-a)=\tan^{-1}(a)\log|\varepsilon/a|+\mathrm{Ti}_2(a)$
$-\frac{1}{2}\log(1+a^2)\tan^{-1}(a)$

(11) $\lim_{x\to\infty}\mathrm{Ti}_2(x,a)=\frac{1}{2}\pi\log x-\frac{1}{4}\pi\log(1+a^2)-\mathrm{Ti}_2(a)+\tan^{-1}(a)\log|a|$

(12) $\mathrm{Ti}_2\left(\frac{2x}{1-x^2},\frac{2y}{1-y^2}\right)=2\mathrm{Ti}_2\left(\frac{x+y}{1-xy},\frac{x+y}{1-xy}\right)+2\mathrm{Ti}_2\left(\frac{1-x}{1+x},\frac{1-x}{1+x}\right)-$
$2\mathrm{Ti}_2(y,y)+\tan^{-1}(x)\log\left[\frac{(1-xy)^2}{(1+x)^2}\cdot\frac{2}{1+y^2}\right]$
$+\tan^{-1}(y)\log\left[\frac{y^2(1+x^2)}{(x+y)^2}\right]-\frac{\pi}{4}\log\left[\frac{2(1-x)^2}{1+x^2}\right]$

(13) $\mathrm{Ti}_2(x,a)+\mathrm{Ti}_2(1/x,1/a)-\mathrm{Ti}_2(x)=\frac{1}{2}\pi\log\left[\frac{a+x}{x(1+a^2)^{\frac{1}{2}}}\right]-\mathrm{Ti}_2(a)$
$+\tan^{-1}(a)\log(a),\qquad x,a>0$

(14) $\mathrm{Ti}_2(-x,a)+\mathrm{Ti}_2(-1/x,1/a)+\mathrm{Ti}_2(x)=-\frac{1}{2}\pi\log\left[\frac{x-a}{x(1+a^2)}\right]$
$-\mathrm{Ti}_2(a)+\tan^{-1}(a)\log(a),\qquad x,a>0$

(15) $\mathrm{Ti}_2(x,1/y)+\mathrm{Ti}_2(y,1/x)=\tan^{-1}(x)\log\left[\frac{1+xy}{(1+y^2)^{1/2}}\right]$
$+\tan^{-1}(y)\log\left[\frac{1+xy}{(1+x^2)^{1/2}}\right]$

(16) $\mathrm{Ti}_2(x,a)-\mathrm{Ti}_2\left(\frac{1-ax}{a+x},a\right)=\tan^{-1}(1/a)\log\left[\frac{a+x}{(1+a^2)^{1/2}}\right]$

(17) $\mathrm{Ti}_2(x,a)-\mathrm{Ti}_2\left(\frac{1-bx}{b+x},b\right)+\mathrm{Ti}_2\left(\frac{1-bx}{b+x},\frac{1+ab}{a-b}\right)-\mathrm{Ti}_2\left(\frac{1}{b},\frac{1+ab}{1-ab}\right)$
$=\tan^{-1}(1/b)\log\left[\frac{(a+x)b}{a(1+b^2)^{1/2}}\right]$

(18) $\mathrm{Ti}_2(x,a)-\mathrm{Ti}_2(x,b)+\mathrm{Ti}_2\left(\frac{1-bx}{b+x},\frac{1+ab}{a-b}\right)-\mathrm{Ti}_2\left(\frac{1}{b},\frac{1+ab}{1-ab}\right)$
$=\tan^{-1}(1/b)\log\left(\frac{a+x}{b+x}\cdot\frac{b}{a}\right)$

Other equations of this nature may be found in Section 3.3.4.

(19) $\mathrm{Ti}_2(\tan\theta,\tan\phi)=\mathrm{Ti}_2[\tan\theta,-\tan(\phi+\theta)]+\theta\log\left[\frac{\sin(\theta+\phi)}{\sin\phi}\right]$

(20) $\mathrm{Ti}_2(\tan\theta,\tan\phi)-\mathrm{Ti}_2(\tan\theta,\tan\psi)=\mathrm{Ti}_2[\tan(\phi-\psi),\tan(\theta+\psi)]$
$-\mathrm{Ti}_2[\tan(\phi-\psi),\tan\psi]+(\phi-\psi)\log\left[\frac{\sin\phi}{\sin(\phi+\theta)}\right]$
$+\theta\log\left[\frac{\sin(\theta+\phi)}{\sin(\theta+\psi)}\right]$

See also Sections 3.3.5 and 3.3.6.

(21) $\mathrm{Ti}_2\left(\frac{a+x}{1-ax}\right)=\mathrm{Ti}_2(a)+\tan^{-1}(a)\log\left[\frac{a+x}{a(1-ax)}\right]+\mathrm{Ti}_2(x,a)$
$+\mathrm{Ti}_2(-x,1/a),\qquad x<1/a$

(22) $\mathrm{Ti}_2\left(\frac{1+ax}{a-x}\right)=\mathrm{Ti}_2(1/a)-\tan^{-1}(1/a)\log\left[\frac{a-x}{a(1+ax)}\right]+\mathrm{Ti}_2(x,1/a)$
$+\mathrm{Ti}_2(-x,a),\qquad x<a$

(23) $\mathrm{Ti}_2\left(\frac{a+x}{1-ax},b\right)=\mathrm{Ti}_2(a,b)+\tan^{-1}(a)\log\left[\frac{a+b+x(1-ab)}{(a+b)(1-ax)}\right]$
$+\mathrm{Ti}_2\left(x,\frac{a+b}{1-ab}\right)+\mathrm{Ti}_2(-x,1/a),\qquad x<1/a$

(24) $\mathrm{Ti}_2\left[\frac{x(a+1/a)}{1-x^2}\right]=\mathrm{Ti}_2(ax)+\mathrm{Ti}_2(x/a)-\mathrm{Ti}_2(xa,a)+\mathrm{Ti}_2(-xa,a)$
$-\mathrm{Ti}_2\left(\frac{x}{a},\frac{1}{a}\right)+\mathrm{Ti}_2\left(-\frac{x}{a},\frac{1}{a}\right),\qquad |x|<1$

Further formulae of this form are given in Sections 3.5.3–3.5.5.

(25) $\mathrm{Ti}_2\left[\frac{x(b+1/b)}{1-x^2},\frac{a(b+1/b)}{1-a^2}\right]-\mathrm{Ti}_2\left[\frac{x(b+1/b)}{1-x^2}\right]=\mathrm{Ti}_2(bx,ab)$
$+\mathrm{Ti}_2(x/b,a/b)+\mathrm{Ti}_2(bx,-b/a)$
$+\mathrm{Ti}_2(x/b,-1/ab)-\mathrm{Ti}_2(bx)-\mathrm{Ti}_2(x/b)$

(26) $\mathrm{Ti}_2\left(\frac{2x}{1-x^2},\frac{2a}{1-a^2}\right)-4\mathrm{Ti}_2(x,a)=\mathrm{Ti}_2\left(\frac{2x}{1-x^2}\right)-2\mathrm{Ti}_2(x)$
$-2\mathrm{Ti}_2\left(\frac{a+x}{1-ax}\right)+2\tan^{-1}(a)\log\left[\frac{a+x}{a(1-ax)}\right]+2\mathrm{Ti}_2(a)$

(27) $$4\,\mathrm{Ti}_2\left(x,\frac{1+a}{1-a}\right)-4\,\mathrm{Ti}_2(x,a)=\mathrm{Ti}_2\left(\frac{2a}{1-a^2}\right)+2\,\mathrm{Ti}_2(a)-2\,\mathrm{Ti}_2\left(\frac{1+a}{1-a}\right)$$
$$-\mathrm{Ti}_2\left[\frac{2x(1-a^2)+2a(1-x^2)}{(1-a^2)(1-x^2)-4ax}\right]+2\,\mathrm{Ti}_2\left[\frac{1+a+x(1-a)}{1-a-x(1+a)}\right]$$
$$-2\,\mathrm{Ti}_2\left(\frac{a+x}{1-ax}\right)+4\tan^{-1}(a)\log\left[\frac{(1+a)(x+a)}{x(1-a)+1+a}\cdot\frac{1}{a}\right]$$
$$-\frac{\pi}{2}\log\left[\frac{1-a}{1+a}\cdot\frac{1+a+x(1-a)}{1-a-x(1+a)}\right]$$

Sections 3.5.7–3.5.10 contain further formulas of a similar character.

(28) $$\tfrac{1}{2}\,\mathrm{Ti}_2\left(\frac{r^2\sin 2\theta}{1-r^2\cos 2\theta}\right)-\tfrac{1}{2}\,\mathrm{Ti}_2\left(\frac{r^2\sin 2\theta}{1-r^2\cos 2\theta},\tan 2\theta\right)$$
$$=\mathrm{Ti}_2\left(\frac{r\sin\theta}{1-r\cos\theta}\right)-\mathrm{Ti}_2\left(\frac{r\sin\theta}{1-r\cos\theta},\tan\theta\right)-\mathrm{Ti}_2\left(\frac{r\sin\theta}{1+r\cos\theta}\right)$$
$$+\mathrm{Ti}_2\left(\frac{r\sin\theta}{1+r\cos\theta},-\tan\theta\right)$$

(29) $$\frac{1}{n}\,\mathrm{Ti}_2(x^n)=\sum_1^n(-1)^{m-1}\,\mathrm{Ti}_2\left[\frac{x-\cos(\theta_m)}{\sin(\theta_m)},\cot(\theta_m)\right]$$
$$+\cos^2(n\pi/2)\{\tfrac{1}{2}\pi\log x+2\sum_1^{[\frac{1}{2}n]}(-1)^{m-1}[\mathrm{Ti}_2(\tan\theta_m)-\theta_m\log(\tan\theta_m)]\},$$
where $\theta_m=(2m-1)\pi/2n$

(30) $$\frac{1}{n}\,\mathrm{Ti}_2[\tan(n\theta),\tan(n\phi)]=\sum_o^{n-1}\left\{\mathrm{Ti}_2\left[\tan\theta,\tan\left(\frac{r\pi-n\phi}{n}\right)\right]\right.$$
$$\left.-\mathrm{Ti}_2\left[\tan\theta,\tan\left(\frac{\overline{2r+1}\pi}{2n}\right)\right]\right\}$$

(31) $$\sum_{x,y,z}\left[\mathrm{Ti}_2\left(\frac{-2x_1x_2}{1+x_1{}^2-x_2{}^2}\right)-\mathrm{Ti}_2\left(\frac{-2x_1x_2}{1+x_1{}^2-x_2{}^2},\frac{2x_1x_2}{x_1{}^2-x_2{}^2}\right)\right.$$
$$\left.-2\,\mathrm{Ti}_2\left(\frac{x_1y_2+x_2y_1}{1+x_2y_2-x_1y_1}\right)+2\,\mathrm{Ti}_2\left(\frac{x_1y_2+x_2y_1}{1+x_2y_2-x_1y_1},\frac{x_1y_2+x_2y_1}{x_1y_1-x_2y_2}\right)\right]=0,$$
where $x_1+y_1+z_1=x_1y_1z_1-x_1y_2z_2-x_2y_2z_1-x_2y_1z_2$,
$x_2+y_2+z_2=x_1y_1z_2+x_1y_2z_1+x_2y_1z_1-x_2y_2z_2$

(32) $$\mathrm{Ti}_2(\tan\theta,\tan\phi)=\frac{1}{2^{2n}}\,\mathrm{Ti}_2[\tan(2^n\theta),\tan(2^n\phi)]-\tfrac{1}{4}\sum_0^{n-1}4^{-m}V(2^m\theta,2^m\phi),$$
where $V(\theta,\phi)=\mathrm{Ti}_2(\tan 2\theta)-2\,\mathrm{Ti}_2(\tan\theta)-2\,\mathrm{Ti}_2[\tan(\theta+\phi)]$
$+2\,\mathrm{Ti}_2(\tan\phi)+2\phi\log[\tan(\theta+\phi)/\tan\phi]$

(33) $$2\,\mathrm{Ti}_2(y,1)=\mathrm{Ti}_2\left(\frac{1-y}{1+y}\right)+\mathrm{Ti}_2(y)$$
$$-\tfrac{1}{2}\,\mathrm{Ti}_2\left(\frac{2y}{1-y^2}\right)-\tfrac{1}{4}\pi\log\left(\frac{1-y}{1+y}\right)-G$$

(34) $$4\,\mathrm{Ti}_2[\tan\theta,\tan(\pi/8)]=2\,\mathrm{Ti}_2[\tan(\theta+\pi/8)]-2\,\mathrm{Ti}_2[\tan(\pi/8)]$$
$$+2\,\mathrm{Ti}_2(\tan\theta)-\tfrac{1}{2}\,\mathrm{Ti}_2(\tan 2\theta)-\tfrac{1}{4}\,\mathrm{Ti}_2(\tan 4\theta)+\tfrac{1}{2}\,\mathrm{Ti}_2[\tan(2\theta+\pi/4)]$$
$$-\tfrac{1}{2}G-\tfrac{1}{4}\pi\log\left[\frac{\tan(\theta+\pi/8)}{\tan(\pi/8)}\right]-\tfrac{1}{8}\pi\log[\tan(2\theta+\pi/4)]$$

For other equations of this nature see Sections 2.3 and 2.7 and Section 3.9.

(35) $$\mathrm{Ti}_2\left(\frac{y+z}{1-yz},A\right)=\mathrm{Ti}_2(y,a)+\mathrm{Ti}_2(y,b)-\mathrm{Ti}_2(y,c)-\mathrm{Ti}_2(y,e)$$
$$+\mathrm{Ti}_2\left(z,\frac{1}{c+e-ce/a}\right)+\mathrm{Ti}_2\left(z,\frac{1}{c+e-ce/b}\right)-\mathrm{Ti}_2(z,1/c)$$
$-\mathrm{Ti}_2(z,1/e)$, where $\tan^{-1}(1/A)=\tan^{-1}(c)+\tan^{-1}(e)-\tan^{-1}(a)$ $-\tan^{-1}(b)$, $z=-y/[ce+y(c+e)]$, and $A(ce-ab)=ab(c+e)$

For a more general relation see Section 3.10.

A.2.4. Clausen's Function

(1) $$\mathrm{Li}_2(e^{i\theta})=\sum_1^\infty\frac{\cos n\theta}{n^2}+i\sum_1^\infty\frac{\sin n\theta}{n^2}=\mathrm{Gl}_2(\theta)+i\,\mathrm{Cl}_2(\theta)$$

(2) $$\mathrm{Gl}_2(\theta)=\frac{\pi^2}{6}+\frac{\theta^2-2\pi|\theta|}{4},\qquad -2\pi\leqq\theta\leqq 2\pi$$

(3) $$\mathrm{Cl}_2(\theta)=-\int_o^\theta\log|2\sin\tfrac{1}{2}\theta|\,d\theta$$

(4) $\mathrm{Cl}_2(2n\pi\pm\theta)=\mathrm{Cl}_2(\pm\theta)=\pm\mathrm{Cl}_2(\theta)$

(5) $\mathrm{Cl}_2(\pi+\theta)=-\mathrm{Cl}_2(\pi-\theta)$

(6) $\mathrm{Cl}_2(\theta)=-\mathrm{Cl}_2(2\pi-\theta)$

(7) $\mathrm{Cl}_2(n\pi)=0$. In particular $\mathrm{Cl}_2(\pi)=0$

(8) $$\int_o^{\pi/2}\log(\sin\theta)\,d\theta=-\tfrac{1}{2}\pi\log 2$$

(9) $\mathrm{Cl}_2(\tfrac{1}{2}\pi)=G$

(10) $\mathrm{Cl}_2(3\pi/2)=-G$

Other numerical relations are given in Sections 4.3 and 4.5.

(11) Maximum value of $\mathrm{Cl}_2(\theta)$ occurs when $\theta=\pi/3$

(12) $\tfrac{1}{2}\mathrm{Cl}_2(2\theta)=\mathrm{Cl}_2(\theta)-\mathrm{Cl}_2(\pi-\theta)$

(13) $$\frac{1}{n}\,\mathrm{Cl}_2(n\theta)=\mathrm{Cl}_2(\theta)+\mathrm{Cl}_2(\theta+2\pi/n)+\cdots+\mathrm{Cl}_2(\theta+\overline{n-1}2\pi/n)$$

(14) $\mathrm{Cl}_2(2\pi/n)+\mathrm{Cl}_2(4\pi/n)+\cdots+\mathrm{Cl}_2(2\overline{n-1}\pi/n)=0$

(15) $\mathrm{Cl}_2(\theta)=-\theta\log|\sin\frac{1}{2}\theta|+\theta+\sum_1^\infty\left(2\theta-2r\pi\log\frac{2r\pi+\theta}{2r\pi-\theta}\right),\ -2\pi<\theta<2\pi$

(16) $\mathrm{Cl}_2(\theta)=\theta\left(1-\log|\theta|+\frac{B_1\theta^2}{2\cdot 32!}+\frac{B_2\theta^4}{4\cdot 54!}+\cdots\right),\qquad 0<|\theta|<2\pi$

(17) $\mathrm{Ti}_2(\tan\theta)=\theta\log|\tan\theta|+\frac{1}{2}\mathrm{Cl}_2(2\theta)+\frac{1}{2}\mathrm{Cl}_2(\pi-2\theta)$

(18) $\mathrm{Ti}_2(\tan\theta,\tan\phi)=\theta\log\left|\frac{\sin(\theta+\phi)}{\cos\theta}\right|+\frac{1}{2}\mathrm{Cl}_2(2\theta+2\phi)+\frac{1}{2}\mathrm{Cl}_2(\pi-2\theta)$
$-\frac{1}{2}\mathrm{Cl}_2(2\phi)$

(19) $\mathrm{Cl}_2(4\phi)=4\,\mathrm{Ti}_2(\tan\phi,\tan\phi)-4\phi\log|2\sin\phi|$

(20) $3\,\mathrm{Cl}_2(\phi)+2\,\mathrm{Cl}_2(\frac{1}{2}\pi-\phi)+\mathrm{Cl}_2(\pi-\phi)=6G,\qquad \tan\phi=4/3$

(21) $\sum_1^5[\mathrm{Cl}_2(2\theta_n)-\mathrm{Cl}_2(2\theta_{n+2}+2\theta_{n+3})+\mathrm{Cl}_2(2\theta_{n+2}+2\theta_{n+3}-2\theta_n)]=0,$
where $\sin(\theta_1+\theta_2+\theta_3+\theta_4+\theta_5)=\sin(\theta_1+\theta_2+\theta_3-\theta_4-\theta_5)+$four other terms cyclically derived (θ_{5+m} is to be interpreted as θ_m)

(22) $\sum_1^5[\mathrm{Cl}_2(2\sigma-2\phi_{n+1}-2\phi_{n+4})+\mathrm{Cl}_2(2\sigma-2\phi_{n+2}-2\phi_{n+3})$
$-\mathrm{Cl}_2(2\phi_n)]=0$, where $\sigma=\frac{1}{2}\sum_1^5\phi_n$ and $\sin\sigma=\sum_1^5\sin(\sigma-2\phi_n)$

In Eqs. (23) to (26) ϕ and θ are related, by $\tan\phi=\frac{1}{3}\tan(\theta/2)$.

(23) $3\,\mathrm{Cl}_2(2\theta)+2\,\mathrm{Cl}_2(2\phi)-3\,\mathrm{Cl}_2(2\phi-\theta)-\mathrm{Cl}_2(3\theta-2\phi)-6\,\mathrm{Cl}_2(\theta)=0,$

(24) $2\,\mathrm{Cl}_2(2\phi)-3\,\mathrm{Cl}_2(2\phi-\theta)-\mathrm{Cl}_2(3\theta-2\phi)-6\,\mathrm{Cl}_2(\pi-\theta)=0,$

(25) $2\,\mathrm{Cl}_2(\pi-\theta)-3\,\mathrm{Cl}_2(2\phi-\theta)+\mathrm{Cl}_2(6\phi-\theta)-6\,\mathrm{Cl}_2(2\phi)=0,$

(26) $8\,\mathrm{Cl}_2(2\phi)-\mathrm{Cl}_2(3\theta-2\phi)-\mathrm{Cl}_2(6\phi-\theta)-8\,\mathrm{Cl}_2(\pi-\theta)=0,$

(27) $2\,\mathrm{Cl}_2(2\phi)+2\,\mathrm{Cl}_2(2\theta)+\mathrm{Cl}_2(4\psi)-\mathrm{Cl}_2(2\phi-2\theta+2\psi)$
$-\mathrm{Cl}_2(2\theta-2\phi+2\psi)-4\,\mathrm{Cl}_2(2\psi)-2\,\mathrm{Cl}_2(2\phi-2\psi)-2\,\mathrm{Cl}_2(2\theta-2\psi)=0,$
where $\sin(\theta-\psi)\sin(\phi-\psi)+\sin\theta\sin\phi=0$

For further equations of this nature see Section 4.6.9.

A.2.5. The Dilogarithm of Complex Argument

(1) $\mathrm{Li}_2(re^{i\theta})=\mathrm{Li}_2(r,\theta)+i[\omega\log(r)+\frac{1}{2}\mathrm{Cl}_2(2\omega)+\frac{1}{2}\mathrm{Cl}_2(2\theta)-\frac{1}{2}\mathrm{Cl}_2(2\omega+2\theta)],$
where $\tan\omega=r\sin\theta/(1-r\cos\theta)$

(2) $\mathrm{Li}_2(r,\theta)=-\frac{1}{2}\int_o^r\frac{\log(1-2r\cos\theta+r^2)}{r}\,dr$

(3) $\mathrm{Li}_2(x,0)=\mathrm{Li}_2(x),\qquad -1\leqq x\leqq 1$

(4) $\mathrm{Li}_2(x,0)=\mathrm{Li}_2(x)+i\pi\log x,\qquad x>1$

(5) $\mathrm{Li}_2(x,\pi)=\mathrm{Li}_2(-x)$

(6) $\mathrm{Li}_2(x,\theta)=\mathrm{Li}_2(x,-\theta)=\mathrm{Li}_2(x,2n\pi\pm\theta)$

(7) $\mathrm{Li}_2(x,\frac{1}{2}\pi)=\mathrm{Li}_2(x,3\pi/2)=\frac{1}{4}\mathrm{Li}_2(-x^2)$

(8) $\mathrm{Li}_2(x,\pi/3)=\frac{1}{6}\mathrm{Li}_2(-x^3)-\frac{1}{2}\mathrm{Li}_2(-x)$

(9) $\mathrm{Li}_2(x,2\pi/3)=\frac{1}{6}\mathrm{Li}_2(x^3)-\frac{1}{2}\mathrm{Li}_2(x)$

(10) $\mathrm{Li}_2(x,\pi/4)=\frac{1}{4}\mathrm{Li}_2(x\sqrt{2-x^2})-\frac{1}{2}\mathrm{Li}_2\left(\frac{-x}{\sqrt{2-x}}\right)+\frac{1}{8}\mathrm{Li}_2\left[\frac{-x^2}{(\sqrt{2-x})^2}\right]$

(11) $\mathrm{Li}_2(x,3\pi/4)=\frac{1}{4}\mathrm{Li}_2(-x\sqrt{2-x^2})-\frac{1}{2}\mathrm{Li}_2\left(\frac{x}{\sqrt{2+x}}\right)$
$+\frac{1}{8}\mathrm{Li}_2\left[\frac{-x^2}{(\sqrt{2+x})^2}\right]$

See also Sections 5.5 and 5.8.

(12) $\mathrm{Li}_2(x,\pi/6)=\frac{1}{12}\mathrm{Li}_2(-x^3)-\frac{1}{4}\mathrm{Li}_2(-x)-\frac{1}{2}\mathrm{Li}_2\left(\frac{-x}{\sqrt{3-x}}\right)$
$+\frac{1}{4}\mathrm{Li}_2(x\sqrt{3-x^2})$

(13) $\mathrm{Li}_2(x,5\pi/6)=\frac{1}{12}\mathrm{Li}_2(x^3)-\frac{1}{4}\mathrm{Li}_2(x)-\frac{1}{2}\mathrm{Li}_2\left(\frac{x}{\sqrt{3+x}}\right)$
$+\frac{1}{4}\mathrm{Li}_2(-x\sqrt{3-x^2})$

(14) $\mathrm{Li}_2(1,\theta)=\frac{1}{4}(\pi-\theta)^2-\pi^2/12,\qquad 0\leqq\theta\leqq 2\pi$

(15) $\mathrm{Li}_2(2\cos\theta,\theta)=(\frac{1}{2}\pi-\theta)^2,\qquad 0\leqq\theta\leqq\pi$

(16) $\mathrm{Li}_2(\cos\theta,\theta)=\frac{1}{4}\mathrm{Li}_2(\cos^2\theta)+\frac{1}{2}(\frac{1}{2}\pi-\theta)^2,\qquad 0\leqq\theta\leqq\pi$

(17) $\mathrm{Li}_2(\sec\theta,\theta)=5\pi^2/24-\frac{1}{4}\mathrm{Li}_2(\cos^2\theta)-\frac{1}{2}\log^2(\cos\theta)-\frac{1}{2}\pi\theta,$
$0\leqq\theta<\frac{1}{2}\pi$

(18) $\mathrm{Li}_2(\frac{1}{2}\sec\theta,\theta)=\pi^2/12-\frac{1}{2}\theta^2-\frac{1}{2}\log^2(2\cos\theta),\qquad -\frac{1}{2}\pi<\theta<\frac{1}{2}\pi$

(19) $\mathrm{Li}_2(\tan\frac{1}{2}\theta,\frac{1}{2}\pi-\theta)=\frac{1}{4}\theta^2+\frac{1}{2}\mathrm{Li}_2(\tan^2\frac{1}{2}\theta)-\frac{1}{4}\mathrm{Li}_2(-\tan^2\frac{1}{2}\theta),$
$-\frac{1}{2}\pi\leqq\theta\leqq\frac{1}{2}\pi$

(20) $\frac{1}{n}\mathrm{Li}_2(x^n,n\theta)=\sum_o^{n-1}\mathrm{Li}_2(x,\theta+2r\pi/n)$

(21) $\frac{1}{2}\mathrm{Li}_2(x^2,2\theta)=\mathrm{Li}_2(x,\theta)+\mathrm{Li}_2(x,\pi-\theta)$

(22) $d\,\mathrm{Li}_2(x,\theta)=-\frac{1}{2}\log(1-2x\cos\theta+x^2)\,dx/x-\tan^{-1}\left(\frac{x\sin\theta}{1-x\cos\theta}\right)d\theta$

(23) $\mathrm{Li}_2(1/x,\theta)+\mathrm{Li}_2(x,\theta)=-\frac{1}{2}\log^2(x)+\frac{1}{2}(\pi-\theta)^2-\pi^2/6,$
$0\leqq\theta\leqq 2\pi$

(24) $\mathrm{Li}_2(x,\theta)+\mathrm{Li}_2(2\cos\theta-x,\theta)=(\frac{1}{2}\pi-\theta)^2+\frac{1}{2}\mathrm{Li}_2(2x\cos\theta-x^2),$
$0\leqq\theta\leqq\pi$

(25) $\mathrm{Li}_2\left(\frac{-x^2}{1-2x\cos\theta}\right)=2\,\mathrm{Li}_2(x,\theta)+2\,\mathrm{Li}_2\left(\frac{-x}{1-2x\cos\theta},\theta\right)$
$+\frac{1}{2}\log^2(1-2x\cos\theta)$

For further examples of single-variable equations, see Section 5.4.

(26) $\mathrm{Li}_2(x,\theta)+\mathrm{Li}_2(y,\theta)+\mathrm{Li}_2(z,\theta)=(\frac{1}{2}\pi-\theta)^2+\frac{1}{2}\mathrm{Li}_2(xy)+\frac{1}{2}\mathrm{Li}_2(yz)$
$+\frac{1}{2}\mathrm{Li}_2(zx)$, where $x+y+z=xyz$

(27) $\mathrm{Li}_2(-a^2,2\alpha)+\mathrm{Li}_2(-b^2,2\beta)+\mathrm{Li}_2(-c^2,2\gamma)=2\,\mathrm{Li}_2(ab,\alpha+\beta)+$
$2\,\mathrm{Li}_2(bc,\beta+\gamma)+2\,\mathrm{Li}_2(ca,\gamma+\alpha)$, where $ae^{i\alpha}+be^{i\beta}+ce^{i\gamma}=$
$abce^{i(\alpha+\beta+\gamma)}$

See Section 5.7 for further details, where a number of formulas of a similar nature are derived.

(28) $\mathrm{Li}_2(ab,\alpha+\beta)=\mathrm{Li}_2(a,\alpha)+\mathrm{Li}_2(b,\beta)+\mathrm{Li}_2\left(b\dfrac{\sin\beta}{\sin\alpha},\pi+\alpha\right)$
$+\mathrm{Li}_2\left(a\dfrac{\sin\alpha}{\sin\beta},\pi+\beta\right)+\frac{1}{2}\log^2\left(\dfrac{a\sin\alpha}{b\sin\beta}\right)$, where $a\sin\alpha-b\sin\beta=$
$ab\sin(\alpha-\beta)$

(29) $\mathrm{Li}_2\left(\dfrac{x}{2\cos\theta-x},2\theta\right)=2\,\mathrm{Li}_2(x,\theta)+\mathrm{Li}_2\left(\dfrac{-x}{2\cos\theta-x}\right)$
$-\frac{1}{2}\mathrm{Li}_2(2x\cos\theta-x^2)$

(30) $\mathrm{Li}_2(x,\theta)=2^{1-n}\,\mathrm{Li}_2(x_n,\theta_n)-\sum_2^n 2^{1-r}[\mathrm{Li}_2(-x_r)-\frac{1}{2}\mathrm{Li}_2(x_{r-1}^2/x_r)]$,
where $\theta_r=2^{r-1}\theta$ and $x_r=\dfrac{x\sin\theta}{\sin\theta_r-x\sin(\theta_r-\theta)}$

(31) $\mathrm{Li}_2(-x,\theta)+\mathrm{Li}_2(-x_1,\phi)-\mathrm{Li}_2(-x_2,\theta+\phi)$
$-\mathrm{Li}_2\left[-\dfrac{\sin(\phi-\theta)}{\sin\theta},\phi\right]+\mathrm{Li}_2\left(-\dfrac{\sin\phi}{\sin\theta},\phi+\theta\right)=\frac{1}{2}\overline{W}(x,\phi)$,
where $x_1=[x\sin\phi+\sin(\phi-\theta)]/\sin\theta$, $x_2=[x\sin(\phi+\theta)+\sin\phi]/\sin\theta$,
$\overline{W}(x,\phi)=\mathrm{Li}_2\left[-\dfrac{x^2\sin\phi+x\sin(\phi-\theta)}{x\sin(\phi+\theta)+\sin\phi}\right]-\mathrm{Li}_2\left[\dfrac{x\sin(\phi+\vartheta)}{\sin\phi+x\sin(\phi+\theta)}\right]$
$+\mathrm{Li}_2\left[1-\dfrac{\sin^2\phi}{\sin^2\theta}\left(1+x\dfrac{\sin(\phi+\theta)}{\sin\phi}\right)\right]-\mathrm{Li}_2\left(1-\dfrac{\sin^2\phi}{\sin^2\theta}\right)$
$+2\log\left[\dfrac{\sin(\phi+\theta)}{\sin\theta}\right]\log\left[\dfrac{x\sin(\phi+\theta)+\sin\phi}{\sin\phi}\right]$

(32) $\mathrm{Li}_2(-x,\theta)-\mathrm{Li}_2\left[\dfrac{x\sin\theta}{x\sin(\theta+\phi)+\sin\phi},\phi\right]$
$+\mathrm{Li}_2\left[-\dfrac{\sin\theta}{x\sin(\theta+\phi)+\sin\phi},\theta+\phi\right]=\mathrm{Li}_2\left[-x\dfrac{\sin(\theta+\phi)}{\sin\phi}\right]$
$+\mathrm{Li}_2\left(-\dfrac{\sin\theta}{\sin\phi},\theta+\phi\right)+\log\left[\dfrac{\sin\phi}{\sin(\theta+\phi)}\right]\log\left[\dfrac{\sin\phi}{x\sin(\theta+\phi)+\sin\phi}\right]$

The follow-up of this equation is given in Section 5.9.8.

(33) $\mathrm{Li}_2\left(-\dfrac{\sin\phi}{\sin\theta},\theta+\phi\right)-\mathrm{Li}_2\left[-\dfrac{\sin(\phi-\theta)}{\sin\theta},\phi\right]=\theta\phi-\frac{1}{2}\mathrm{Li}_2\left(\dfrac{\sin^2\theta}{\sin\phi}\right)$
$+\log\left(\dfrac{\sin\phi}{\sin\theta}\right)\log\left[\dfrac{\sin(\phi-\theta)}{\sin\phi}\right]$

(34) $\mathrm{Li}_2\left[\dfrac{\sin\theta}{\sin(\theta+\phi)},\phi\right]+\mathrm{Li}_2\left(-\dfrac{\sin\theta}{\sin\phi},\theta+\phi\right)$
$=\frac{1}{2}\theta^2-\frac{1}{2}\log^2\left[\dfrac{\sin(\theta+\phi)}{\sin\phi}\right]$

(35) $\mathrm{Li}_2\left[-\dfrac{\sin(\phi+N\theta)}{\sin\theta},\phi+\overline{N+1}\theta\right]-\mathrm{Li}_2\left(-\dfrac{\sin\phi}{\sin\theta},\phi+\theta\right)$
$=N\theta[\phi+\frac{1}{2}(N+1)\theta]-\frac{1}{2}\sum_1^N\mathrm{Li}_2\left[\dfrac{\sin^2\theta}{\sin^2(\phi+r\theta)}\right]$
$+\sum_1^N\log\left[\dfrac{\sin(\phi+r\theta)}{\sin\theta}\right]\log\left[\dfrac{\sin(\phi+\overline{r-1}\theta)}{\sin(\phi+r\theta)}\right]$, $\quad 0<\phi+N\theta<\pi$

(36) $\mathrm{Li}_2\left(-\dfrac{\sin N\theta}{\sin\theta},\overline{N+1}\theta\right)=\frac{1}{2}\theta^2N(N+1)-\frac{1}{2}\sum_1^N\mathrm{Li}_2\left(\dfrac{\sin^2\theta}{\sin^2 r\theta}\right)$
$+\sum_2^N\log\left(\dfrac{\sin r\theta}{\sin\theta}\right)\log\left(\dfrac{\sin\overline{r-1}\theta}{\sin r\theta}\right)$

(37) $\sum_2^{M-1}\mathrm{Li}_2\left[\dfrac{\sin^2(\pi/M)}{\sin^2(r\pi/M)}\right]$
$=2\sum_2^{M-1}\log\left[\dfrac{\sin(r\pi/M)}{\sin(\pi/M)}\right]\log\left[\dfrac{\sin(\overline{r-1}\pi/M)}{\sin(r\pi/M)}\right]+\pi^2(\frac{1}{2}-1/M)$

(38) $M\,\mathrm{Li}_2(-x,\theta)=\mathrm{Li}_2(-x_M,M\theta)-(M-1)(\frac{1}{2}M\theta^2-\pi^2/12)$
$+\frac{1}{2}\sum_{n=1}^{M-1}W(x,n\theta)$, $\quad -\pi<M\theta<\pi$, where
$x_M=[x\sin(M\theta)+\sin(\overline{M-1}\theta)]/\sin\theta$ and
$W(x,n\theta)=\mathrm{Li}_2\left[-\dfrac{x^2\sin n\theta+x\sin(\overline{n-1}\theta)}{x\sin(\overline{n+1}\theta)+\sin(n\theta)}\right]$
$-\mathrm{Li}_2\left[\dfrac{x\sin(\overline{n+1}\theta)}{x\sin(\overline{n+1}\theta)+\sin n\theta}\right]$
$+\mathrm{Li}_2\left\{1-\dfrac{\sin^2(n\theta)}{\sin^2\theta}\left[1+x\dfrac{\sin(\overline{n+1}\theta)}{\sin(n\theta)}\right]\right\}+2\log\left[\dfrac{\sin(\overline{n+1}\theta)}{\sin\theta}\right]$
$\times\log\left[\dfrac{x\sin(\overline{n+1}\theta)+\sin(n\theta)}{\sin\theta}\right]$

(39) $\mathrm{Li}_2(-x, \pi/M) = \dfrac{(M-3)(M-2)}{12M^2}\pi^2 + \dfrac{1}{2M}\sum_{n=1}^{M-1} W(x, n\pi/M)$ with W as in (38) above.

(40) $\mathrm{Li}_2(-x_r, r\pi/M) = r\,\mathrm{Li}_2(-x, \pi/M) + (r-1)\left(\dfrac{r}{2M^2} - \dfrac{1}{12}\right)\pi^2$

$-\frac{1}{2}\sum_{n=1}^{r-1} W(x, n\pi/M)$, where W is as in (38) above,

and $x_r = [x\sin(r\pi/M) + \sin(\overline{r-1}\,\pi/M)]/\sin(\pi/M)$

Other equations of a similar character to the above can be found in section 5.9.

A.2.6. The Trilogarithm

(1) $\mathrm{Li}_3(z) = \int_o^z \dfrac{\mathrm{Li}_2(z)}{z}\,dz = \dfrac{z}{1^3} + \dfrac{z^2}{2^3} + \cdots, \qquad |z| \leqq 1$

(2) $\mathrm{Li}_3(-1) = -\frac{3}{4}\mathrm{Li}_3(1) = -\frac{3}{4}\zeta(3)$

(3) $\mathrm{Li}_3(\frac{1}{2}) = \frac{7}{8}\mathrm{Li}_3(1) - \dfrac{\pi^2}{12}\log 2 + \frac{1}{6}\log^3(2)$

For other numerical relations see Section 6.3.

(4) $\frac{1}{4}\mathrm{Li}_3(x^2) = \mathrm{Li}_3(x) + \mathrm{Li}_3(-x)$

(5) $\mathrm{Li}_3(-x) - \mathrm{Li}_3(-1/x) = -\dfrac{\pi^2}{6}\log x - \frac{1}{6}\log^3(x)$

(6) $\mathrm{Li}_3(x) - \mathrm{Li}_3(1/x) = \dfrac{\pi^2}{3}\log x - \frac{1}{6}\log^3(x) - \frac{1}{2}i\pi\log^2(x), \qquad x > 1$

(7) $\mathrm{Li}_3\left(\dfrac{-x}{1-x}\right) + \mathrm{Li}_3(1-x) + \mathrm{Li}_3(x) = \mathrm{Li}_3(1) + \dfrac{\pi^2}{6}\log(1-x)$

$-\frac{1}{2}\log(x)\log^2(1-x) + \frac{1}{6}\log^3(1-x), \qquad 0 < x < 1$

(8) $\mathrm{Li}_3\left(\dfrac{x}{1+x}\right) + \mathrm{Li}_3\left(\dfrac{1}{1+x}\right) + \mathrm{Li}_3(-x) = \mathrm{Li}_3(1) - \dfrac{\pi^2}{6}\log(1+x)$

$-\frac{1}{2}\log(x)\log^2(1+x) + \frac{1}{3}\log^3(1+x), \qquad x > 0$

(9) $\mathrm{Li}_3(1-1/x) + \mathrm{Li}_3(1-x) + \mathrm{Li}_3(x) = \mathrm{Li}_3(1) + \dfrac{\pi^2}{6}\log x + \frac{1}{6}\log^3(x)$

$-\frac{1}{2}\log^2(x)\log(1-x), \qquad 0 < x < 1$

(10) $\mathrm{Li}_3\left(\dfrac{1-x}{1+x}\right) - \mathrm{Li}_3\left(\dfrac{x-1}{x+1}\right) = 2\mathrm{Li}_3(1-x) + 2\mathrm{Li}_3\left(\dfrac{1}{1+x}\right)$

$-\frac{1}{2}\mathrm{Li}_3(1-x^2) - \frac{7}{4}\mathrm{Li}_3(1) + \dfrac{\pi^2}{6}\log(1+x) - \frac{1}{3}\log^3(1+x)$

For other single-variable equations see Sections 6.4 and 6.7.

(11) $\mathrm{Li}_3\left[\dfrac{x}{y}\dfrac{(1-y)^2}{(1-x)^2}\right] + \mathrm{Li}_3(xy) + \mathrm{Li}_3(x/y) - 2\mathrm{Li}_3\left[\dfrac{x(1-y)}{y(1-x)}\right]$

$-2\mathrm{Li}_3\left[\dfrac{x(1-y)}{x-1}\right] - 2\mathrm{Li}_3\left(\dfrac{1-y}{1-x}\right) - 2\mathrm{Li}_3\left[\dfrac{1-y}{y(x-1)}\right] - 2\mathrm{Li}_3(x)$

$-2\mathrm{Li}_3(y) + 2\mathrm{Li}_3(1) = \log^2(y)\log\left(\dfrac{1-y}{1-x}\right) - \dfrac{\pi^2}{3}\log y - \frac{1}{3}\log^3(y)$

Further two-variable functional equations are given in Sections 6.7 and 6.8 and in Section 7.7.

(12) $\Lambda_3(x) = \int_o^x \dfrac{\log^2|x|}{1+x}\,dx$

(13) $\mathrm{Li}_3(-x) = \log|x|\,\mathrm{Li}_2(-x) + \frac{1}{2}\log^2|x|\log(1+x) - \frac{1}{2}\Lambda_3(x)$

Equations involving Λ_3 may be found in Section 6.7.

(14) $\mathrm{Ti}_3(y) = \int_o^y \dfrac{\mathrm{Ti}_2(y)}{y}\,dy = \dfrac{y}{1^3} - \dfrac{y^3}{3^3} + \dfrac{y^5}{5^3} - \cdots, \qquad |y| \leqq 1$

(15) $\mathrm{Ti}_3(y) + \mathrm{Ti}_3(1/y) = \mathrm{sgn}(y)(\pi^3/16 + \frac{1}{4}\pi\log^2|y|)$

(16) $\mathrm{Ti}_3(1) = \pi^3/32$

(17) $\mathrm{Ti}_3(\rho, a) = \int_o^\rho \left[\dfrac{\mathrm{Ti}_2(\rho)}{\rho+a} + \dfrac{\mathrm{Ti}_2(\rho, a)}{\rho} - \dfrac{\mathrm{Ti}_2(\rho, a)}{\rho+a}\right]d\rho$

(18) $\mathrm{Li}_3(re^{i\theta}) = \mathrm{Li}_3(r, \theta) + i[\mathrm{Ti}_3(\rho) - \mathrm{Ti}_3(\rho, \tan\theta)], \qquad \rho = \dfrac{r\sin\theta}{1 - r\cos\theta}$

(19) $\mathrm{Li}_3(e^{i\theta}) = \mathrm{Cl}_3(\theta) + i\,\mathrm{Gl}_3(\theta)$

(20) $\mathrm{Cl}_3(\theta) = \sum_1^\infty \dfrac{\cos n\theta}{n^3} = \mathrm{Li}_3(1) - \int_o^\theta \mathrm{Cl}_2(\theta)\,d\theta$

(21) $\mathrm{Gl}_3(\theta) = \theta(\pi - |\theta|)(2\pi - |\theta|)/12, \qquad -2\pi \leqq \theta \leqq 2\pi$

(22) $\mathrm{Cl}_3(0) = \mathrm{Li}_3(1) = \zeta(3)$

(23) $\mathrm{Cl}_3(\pi) = \mathrm{Li}_3(-1) = -\frac{3}{4}\mathrm{Li}_3(1)$

(24) $\mathrm{Cl}_3(\frac{1}{2}\pi) = -\frac{3}{32}\mathrm{Li}_3(1)$

(25) $\mathrm{Cl}_3(\frac{1}{3}\pi) = \frac{1}{3}\mathrm{Li}_3(1)$

(26) $\mathrm{Cl}_3(\frac{2}{3}\pi) = -\frac{4}{9}\mathrm{Li}_3(1)$

(27) $\mathrm{Cl}_3(2n\pi \pm \theta) = \mathrm{Cl}_3(\theta)$

(28) $\mathrm{Cl}_3(\pi + \theta) = \mathrm{Cl}_3(\pi - \theta)$

(29) $\dfrac{1}{r^2}\mathrm{Cl}_3(r\theta) = \mathrm{Cl}_3(\theta) + \mathrm{Cl}_3(\theta + 2\pi/r) + \cdots + \mathrm{Cl}_3(\theta + 2\pi\overline{r-1}/r)$

(30) $\mathrm{Cl}_3(2\pi/r) + \mathrm{Cl}_3(4\pi/r) + \cdots + \mathrm{Cl}_3(2\pi\overline{r-1}/r) = -(1 - 1/r^2)\mathrm{Li}_3(1)$

(31) $\mathrm{Li}_3(iy) = \frac{1}{8}\mathrm{Li}_3(-y^2) + i\,\mathrm{Ti}_3(y)$

(32) $\dfrac{1}{n^2}\mathrm{Li}_3(r^n, n\theta) = \mathrm{Li}_3(r, \theta) + \mathrm{Li}_3(r, \theta + 2\pi/n) + \cdots$

$+ \mathrm{Li}_3(r, \theta + 2\pi\overline{n-1}/n)$

(33) $\frac{1}{4}\,\mathrm{Li}_3(r^2,2\theta)=\mathrm{Li}_3(r,\theta)+\mathrm{Li}_3(r,\theta+\pi)=\mathrm{Li}_3(r,\theta)+\mathrm{Li}_3(r,\pi-\theta)$

(34) $\mathrm{Li}_3(r,\theta)-\mathrm{Li}_3\left(\frac{1}{r},\theta\right)=-\frac{1}{6}\log^3(r)+\frac{1}{6}\log(r)[3(\theta-\pi)^2-\pi^2]$,

$0<\theta<2\pi$

(35) $\mathrm{Li}_3(r,\pi)=\frac{1}{4}\,\mathrm{Li}_3(r^2)-\mathrm{Li}_3(r)$

(36) $\mathrm{Li}_3(r,\frac{1}{2}\pi)=\frac{1}{8}\,\mathrm{Li}_3(-r^2)$

(37) $\mathrm{Li}_3(r,\pi/3)=\frac{1}{18}\,\mathrm{Li}_3(-r^3)-\frac{1}{2}\,\mathrm{Li}_3(-r)$

(38) $\mathrm{Li}_3(r,2\pi/3)=\frac{1}{18}\,\mathrm{Li}_3(r^3)-\frac{1}{2}\,\mathrm{Li}_3(r)$

(39) $\mathrm{Ls}_3(\theta)=-\int_o^\theta \log^2(2\sin\frac{1}{2}\theta)\,d\theta$

(40) $\mathrm{Ls}_3(\pi)=\frac{1}{2}\,\mathrm{Ls}_3(2\pi)=-\pi^3/12$

(41) $\mathrm{Ls}_3(\pi/3)=-7\pi^3/108$

(42) $\int_o^\pi \theta\log(2\sin\frac{1}{2}\theta)\,d\theta=\frac{7}{4}\,\mathrm{Li}_3(1)$

(43) $\int_o^{\frac{1}{2}\pi} \theta\log(2\sin\frac{1}{2}\theta)\,d\theta=\frac{35}{32}\,\mathrm{Li}_3(1)-\frac{1}{2}\pi G$

(44) $\mathrm{Ti}_3(\tan\frac{1}{2}\theta)=\mathrm{Ti}_2(\tan\frac{1}{2}\theta)\log(\tan\frac{1}{2}\theta)-\frac{1}{4}\theta\log^2(\tan\frac{1}{2}\theta)+\pi^3/24$
$+\frac{1}{2}\,\mathrm{Ls}_3(\pi-\theta)-\frac{1}{2}\,\mathrm{Ls}_3(\theta)+\frac{1}{8}\,\mathrm{Ls}_3(2\theta)$

(45) $\mathrm{Li}_3(r,\theta)=\mathrm{Li}_3(r)+\frac{1}{2}\int_o^\theta(\theta-\phi)\log(1-2r\cos\phi+r^2)\,d\phi$

(46) $\mathrm{Re}\,\mathrm{Li}_3[\frac{1}{2}(1-e^{i2\theta})]=\mathrm{Re}\,\mathrm{Li}_3[\sin\theta e^{i(\theta-\pi/2)}]=\frac{7}{16}\,\mathrm{Li}_3(1)+\frac{1}{8}\,\mathrm{Li}_3(\sin^2\theta)$
$+\frac{1}{2}\theta^2\log(\sin\theta)-\frac{1}{4}\,\mathrm{Cl}_3(2\theta)+\frac{1}{4}\,\mathrm{Cl}_3(\pi-2\theta)$

(47) $\mathrm{Re}\,\mathrm{Li}_3[\tan\theta e^{i(\pi/2-2\theta)}]=\frac{5}{16}\,\mathrm{Li}_3(1)+\frac{1}{4}\,\mathrm{Li}_3(\tan^2\theta)-\frac{1}{8}\,\mathrm{Li}_3(-\tan^2\theta)$
$+\theta^2\log(\tan\theta)+\frac{1}{4}\,\mathrm{Cl}_3(\pi-4\theta)-\frac{1}{8}\,\mathrm{Cl}_3(4\theta)$

(48) $\mathrm{Re}\,\mathrm{Li}_3(1-e^{i\theta})=\mathrm{Re}\,\mathrm{Li}_2[2\sin(\frac{1}{2}\theta)e^{i(\frac{1}{2}\theta-\pi/2)}]=\frac{1}{2}\,\mathrm{Li}_3(1)-\frac{1}{2}\,\mathrm{Cl}_3(\theta)$
$+\frac{1}{4}\theta^2\log(2\sin\frac{1}{2}\theta)$

(49) $\mathrm{Im}\,\mathrm{Li}_3(1-e^{i\theta})=\theta^3/24-\frac{1}{2}\theta\log^2|2\sin\frac{1}{2}\theta|-\mathrm{Cl}_2(\theta)\log|2\sin\frac{1}{2}\theta|+\mathrm{Ls}_3(\theta)$

(50) $\mathrm{Li}_3\left(\frac{\sin\phi}{\sin\theta},\phi-\theta\right)+\mathrm{Li}_3\left(-\frac{\sin\phi}{\sin\theta},\phi+\theta\right)=\frac{1}{2}\phi^2\log\left(\frac{\sin\phi}{\sin\theta}\right)^2$
$+\frac{1}{4}\,\mathrm{Li}_3\left(\frac{\sin^2\phi}{\sin^2\theta}\right)+\frac{1}{4}\,\mathrm{Cl}_3(2\theta+2\phi)+\frac{1}{4}\,\mathrm{Cl}_3(2\theta-2\phi)-\frac{1}{2}\,\mathrm{Cl}_3(2\theta)$
$-\frac{1}{2}\,\mathrm{Cl}_3(2\phi)+\frac{1}{2}\,\mathrm{Li}_3(1)$

(51) $\mathrm{Li}_3(x,2\psi)-2\,\mathrm{Li}_3\left(\frac{2x\sin\psi}{1-x},\frac{1}{2}\pi+\psi\right)-2\,\mathrm{Li}_3\left(\frac{2\sin\psi}{1-x},\frac{1}{2}\pi-\psi\right)$
$=\mathrm{Li}_3(x)-\mathrm{Li}_3(1)+\mathrm{Cl}_3(2\psi)-\frac{1}{2}\,\mathrm{Li}_3\left(\frac{-4x\sin^2\psi}{(1-x)^2}\right)-2\psi^2\log\left(\frac{2\sin\psi}{1-x}\right)$

(52) $\frac{1}{2}\,\mathrm{Cl}_3(4\alpha)+\frac{1}{2}\,\mathrm{Cl}_3(2\theta+2\phi)+\frac{1}{2}\,\mathrm{Cl}_3(2\theta-2\phi)-\mathrm{Cl}_3(\phi-\theta+2\alpha)$
$-\mathrm{Cl}_3(\phi-\theta-2\alpha)-\mathrm{Cl}_3(\phi+\theta+\pi+2\alpha)-\mathrm{Cl}_3(\phi+\theta+\pi-2\alpha)-\mathrm{Cl}_3(2\theta)$
$-\mathrm{Cl}_3(2\phi)+\mathrm{Cl}_3(0)=2i\alpha(\theta^2-\phi^2)$, where $\sin\phi/\sin\theta=e^{2i\alpha}$

(53) $\mathrm{Ls}_3(2\omega,2\theta)=2\,\mathrm{Ti}_3(\tan\omega)-2\,\mathrm{Ti}_3(\tan\omega,\tan\theta)+\frac{1}{2}\,\mathrm{Ls}_3(2\omega)$
$+\frac{1}{2}\,\mathrm{Ls}_3(2\omega+2\theta)-\frac{1}{2}\,\mathrm{Ls}_3(2\theta)+\log\left[\frac{\sin(\omega+\theta)}{\sin\omega}\right]$
$\times\left[\mathrm{Cl}_2(2\omega)+\mathrm{Cl}_2(2\theta)-\mathrm{Cl}_2(2\theta+2\omega)-\omega\log\frac{\sin(\omega+\theta)}{\sin\omega}\right]$

A.2.7. The Higher-Order Functions

(1) $\mathrm{Li}_n(z)=\int_o^z\frac{\mathrm{Li}_{n-1}(z)}{z}\,dz=\frac{z}{1^n}+\frac{z^2}{2^n}+\cdots,\qquad |z|\leqq 1$

(2) $\mathrm{Li}_n(-1)=-(1-2^{1-n})\,\mathrm{Li}_n(1)$

(3) $\mathrm{Li}_{2n}(1)=2^{2n-1}B_n\pi^{2n}/(2n)!$

(4) $\mathrm{Li}_{2n}(-1)=-(2^{2n-1}-1)B_n\pi^{2n}/(2n)!$

Section 7.2 gives the first eight Bernoulli numbers.

(5) $\frac{1}{2^{n-1}}\,\mathrm{Li}_n(x^2)=\mathrm{Li}_n(x)+\mathrm{Li}_n(-x)$

(6) $\mathrm{Li}_n(-x)+(-1)^n\,\mathrm{Li}_n(-1/x)=-\frac{1}{n!}\log^n(x)+$
$2\sum_{r=1}^{[n/2]}\log^{n-2r}(x)\,\mathrm{Li}_{2r}(-1)/(n-2r)!$

(7) $\Lambda_n(x)=\int_o^x\frac{\log^{n-1}|x|}{1+x}\,dx$

(8) $\mathrm{Li}_n(x)=\log(x)\,\mathrm{Li}_{n-1}(x)-\frac{1}{2!}\log^2(x)\,\mathrm{Li}_{n-2}(x)+\cdots$
$+\frac{(-1)^{n-1}}{(n-2)!}\log^{n-2}(x)\,\mathrm{Li}_2(x)+\frac{(-1)^{n-1}}{(n-1)!}\log^{n-1}(x)\log(1-x)$
$+\frac{(-1)^n}{(n-1)!}\,\Lambda_n(-x)$

(9) $\frac{1}{2^{n-1}}\,\Lambda_n(x^2)=\Lambda_n(x)+\Lambda_n(-x)$

(10) $\Lambda_n(x)+(-1)^n\Lambda_n(1/x)=\frac{1}{n}\log^n|x|+[1+(-1)^n]\Lambda_n(\pm 1)$ according to the sign of x

(11) $\mathrm{Ti}_n(y)=\int_o^y\frac{\mathrm{Ti}_{n-1}(y)}{y}\,dy=\frac{y}{1^n}-\frac{y^3}{3^n}+\frac{y^5}{5^n}-\cdots,\qquad |y|\leqq 1$

(12) $\mathrm{Ti}_n(y)+(-1)^{n-1}\,\mathrm{Ti}_n(1/y)=\mathrm{sgn}(y)\left[\frac{\frac{1}{2}\pi\log^{n-1}|y|}{(n-1)!}\right.$
$\left.+2\sum_{r=1}^{[(n-1)/2]}\frac{\log^{n-1-2r}|y|}{(n-1-2r)!}\,\mathrm{Ti}_{2r+1}(1)\right]$

(13) $\mathrm{Ti}_{2n+1}(1)=E_n\pi^{2n+1}/(2^{2n+2}(2n)!)$

The first five Euler numbers are given in Section 7.2.

(14) $\mathrm{Ti}_n(\rho, a)=\int_o^\rho\left[\frac{\mathrm{Ti}_{n-1}(\rho)}{\rho+a}+\frac{\mathrm{Ti}_{n-1}(\rho,a)}{\rho}-\frac{\mathrm{Ti}_{n-1}(\rho,a)}{\rho+a}\right]d\rho$

(15) $\mathrm{Li}_n(re^{i\theta})=\mathrm{Li}_n(r,\theta)+i[\mathrm{Ti}_n(\rho)-\mathrm{Ti}_n(\rho,\tan\theta)], \qquad \rho=\frac{r\sin\theta}{1-r\cos\theta}$

(16) $\mathrm{Li}_{2n}(e^{i\theta})=\mathrm{Gl}_{2n}(\theta)+i\,\mathrm{Cl}_{2n}(\theta)$

(17) $\mathrm{Li}_{2n+1}(e^{i\theta})=\mathrm{Cl}_{2n+1}(\theta)+i\,\mathrm{Gl}_{2n+1}(\theta)$

(18) $\mathrm{Cl}_{2n}(\theta)=\sum_1^\infty\frac{\sin r\theta}{r^{2n}}=\int_o^\theta\mathrm{Cl}_{2n-1}(\theta)\,d\theta$

(19) $\mathrm{Cl}_{2n+1}(\theta)=\sum_1^\infty\frac{\cos r\theta}{r^{2n+1}}=\mathrm{Li}_{2n+1}(1)-\int_o^\theta\mathrm{Cl}_{2n}(\theta)\,d\theta$

(20) $\mathrm{Gl}_{2n}(\theta)=\sum_1^\infty\frac{\cos r\theta}{r^{2n}}$

(21) $\mathrm{Gl}_{2n+1}(\theta)=\sum_1^\infty\frac{\sin r\theta}{r^{2n+1}}$

(22) $\mathrm{Gl}_n(\theta)=(-1)^{1+[n/2]}2^{n-1}\pi^nB_n[\theta/2\pi]/n!, \qquad 0\leqq\theta\leqq 2\pi,\ n>1$

(23) $\frac{1}{m^{n-1}}\mathrm{Cl}_n(m\theta)=\sum_{r=o}^{m-1}\mathrm{Cl}_n(\theta+2\pi r/m)$

(24) $\frac{1}{2^{2n}}\mathrm{Cl}_{2n+1}(2\theta)=\mathrm{Cl}_{2n+1}(\theta)+\mathrm{Cl}_{2n+1}(\pi-\theta)$

(25) $\mathrm{Cl}_{2n+1}(\frac{1}{2}\pi)=-2^{-(2n+1)}(1-2^{-2n})\mathrm{Li}_{2n+1}(1)$

(26) $\mathrm{Cl}_{2n+1}(\pi/3)=\frac{1}{2}(1-2^{-2n})(1-3^{-2n})\mathrm{Li}_{2n+1}(1)$

Other formulas involving Cl_n and Gl_n can be found in Sections 7.2, 7.3, and 7.5.

(27) $\mathrm{Li}_n(iy)=\frac{1}{2^n}\mathrm{Li}_n(-y^2)+i\,\mathrm{Ti}_n(y)$

(28) $\frac{1}{m^{n-1}}\mathrm{Li}_n(r^m, m\theta)=\sum_{l=o}^{m-1}\mathrm{Li}_n(r,\theta+2\pi l/m)$

(29) $\mathrm{Li}_n(r,\theta)+(-1)^n\mathrm{Li}_n(1/r,\theta)=-\frac{1}{n!}\log^n(r)$
$+2\sum_{m=1}^{[n/2]}\frac{\log^{n-2m}(r)}{(n-2m)!}\mathrm{Gl}_{2m}(\theta)$

(30) $\mathrm{Li}_n(-r,\theta)=\mathrm{Li}_n(r,\pi-\theta)$

(31) $\mathrm{Li}_n(r,0)=\mathrm{Li}_n(r), \qquad r<1$

(32) $\mathrm{Li}_n(r,\pi)=\frac{1}{2^{n-1}}\mathrm{Li}_n(r^2)-\mathrm{Li}_n(r)$

(33) $\mathrm{Li}_n(r,\frac{1}{2}\pi)=\frac{1}{2^n}\mathrm{Li}_n(-r^2)$

(34) $\mathrm{Li}_n(r,\pi/3)=\frac{1}{2}\cdot\frac{1}{3^{n-1}}\mathrm{Li}_n(-r^3)-\frac{1}{2}\mathrm{Li}_n(-r)$

(35) $\mathrm{Re}\,\mathrm{Li}_4(1-e^{i\theta})=\frac{1}{4}\mathrm{Ls}_4^{(1)}(\theta)-\frac{1}{4}\theta\,\mathrm{Ls}_3(\theta)+\frac{\theta^2}{8}\log^2(2\sin\frac{1}{2}\theta)$
$+\frac{1}{2}[\mathrm{Li}_3(1)-\mathrm{Cl}_3(\theta)]\log(2\sin\frac{1}{2}\theta)-\theta^4/192$

(36) $\mathrm{Im}\,\mathrm{Li}_4(1-e^{i\theta})=-\frac{1}{6}\mathrm{Ls}_4(\theta)+\frac{1}{2}\mathrm{Ls}_3(\theta)\log(2\sin\frac{1}{2}\theta)-\frac{\theta}{6}\log^3(2\sin\frac{1}{2}\theta)$
$+\frac{3}{2}\mathrm{Cl}_2(\theta)\log^2(2\sin\frac{1}{2}\theta)+\frac{\theta^3}{24}\log(2\sin\frac{1}{2}\theta)-\frac{1}{4}\mathrm{Cl}_4(\theta)+\frac{1}{4}\theta\,\mathrm{Li}_3(1)$

(37) $\int_o^\pi\log^3(2\sin\frac{1}{2}\theta)=\frac{3\pi}{2}\mathrm{Li}_3(1)$

(38) $\int_o^{\pi/3}\theta\log^2(2\sin\frac{1}{2}\theta)\,d\theta=17\pi^4/6480$

Other numerical values of log-sine integrals can be found in Sections 7.6 and 7.9.

(39) $\frac{1}{8}\mathrm{Li}_4(x^2)=\mathrm{Li}_4(x)+\mathrm{Li}_4(-x)$

(40) $\mathrm{Li}_4(-x)+\mathrm{Li}_4(-1/x)=-7\pi^4/360-\frac{1}{24}\log^4(x)-\frac{1}{12}\pi^2\log^2(x)$

(41) $\mathrm{Li}_4(-x^2y\eta/\xi)+\mathrm{Li}_4(-y^2x\xi/\eta)+\mathrm{Li}_4(x^2y/\eta^2\xi)+\mathrm{Li}_4(y^2x/\xi^2\eta)$
$=6\,\mathrm{Li}_4(xy)+6\,\mathrm{Li}_4(xy/\xi\eta)+6\,\mathrm{Li}_4(-xy/\eta)+6\,\mathrm{Li}_4(-xy/\xi)$
$+3\,\mathrm{Li}_4(x\eta)+3\,\mathrm{Li}_4(y\xi)+3\,\mathrm{Li}_4(x/\eta)+3\,\mathrm{Li}_4(y/\xi)$
$+3\,\mathrm{Li}_4(-x\eta/\xi)+3\,\mathrm{Li}_4(-y\xi/\eta)+3\,\mathrm{Li}_4(-x/\eta\xi)+3\,\mathrm{Li}_4(-y/\eta\xi)$
$-6\,\mathrm{Li}_4(x)-6\,\mathrm{Li}_4(y)-6\,\mathrm{Li}_4(-x/\xi)-6\,\mathrm{Li}_4(-y/\eta)+\frac{3}{2}\log^2(\xi)\log^2(\eta)$,
where $\xi=1-x$, $\eta=1-y$

Single-variable formulas derived from (41), together with similar formulas for $\Lambda_4(x)$, will be found in Section 7.7.

(42) $\frac{1}{16}\mathrm{Li}_5(x^2)=\mathrm{Li}_5(x)+\mathrm{Li}_5(-x)$

(43) $\mathrm{Li}_5(-x)-\mathrm{Li}_5(-1/x)=-\frac{1}{120}\log^5(x)-\frac{1}{36}\pi^2\log^3(x)-\frac{7}{360}\pi^4\log(x)$

(44) $\mathrm{Li}_5(-x^3)+\mathrm{Li}_5(-\xi^3)+\mathrm{Li}_5(x^3/\xi^3)=9\,\mathrm{Li}_5(x\xi)+9\,\mathrm{Li}_5(-x/\xi^2)$
$+9\,\mathrm{Li}_5(-x^2/\xi)+54\,\mathrm{Li}_5(x)+54\,\mathrm{Li}_5(\xi)+54\,\mathrm{Li}_5(-x/\xi)$
$+81\,\mathrm{Li}_5(-x)+81\,\mathrm{Li}_5(-\xi)+81\,\mathrm{Li}_5(x/\xi)+\frac{9}{4}\log^4(\xi)\log x$
$+\frac{19}{120}\pi^4\log(\xi)-\frac{9}{4}\pi^2\log^3(\xi)-\frac{63}{40}\log^5(\xi)+21\,\mathrm{Li}_5(1)$, where $\xi=1-x$

Special cases of this formula, together with a two-variable functional equation for $\Lambda_5(x)$, will be found in Section 7.8.

(45) $\mathrm{Ls}_{n+1}(\pi)=-\pi\left(\frac{d}{dx}\right)^n\exp\left[\sum_2^\infty\frac{x^n}{n}(-1)^n(1-2^{1-n})\zeta(n)\right]_{x=0}$

(46) $\mathrm{Ls}_{m+2}(\pi)=(-1)^m m![\pi(1-2^{-m})\zeta(m+1)$
$-(1-2^{2-m})\zeta(m-1)\mathrm{Ls}_3(\pi)/2!+(1-2^{3-m})\zeta(m-2)\mathrm{Ls}_4(\pi)/3!+\cdots$
$+(-1)^m(1-\frac{1}{2})\zeta(2)\mathrm{Ls}_m(\pi)/(m-1)!]$

(47) $\int_o^\pi \log^n[(2\sin\frac{1}{2}\theta)(2\cos\frac{1}{2}\theta)^p]\,d\theta$

$$=-\pi\left(\frac{d}{dx}\right)^n \exp\left[\sum_2^\infty \frac{x^n}{n}(-1)^n\left(1+p^n-\left(\tfrac{1}{2}\right)^n-\left(\tfrac{1}{2}p\right)^n-\left(\tfrac{1}{2}\overline{1+p}\right)^n\right)\zeta(n)\right]_{x=0}$$

Many other formulas of this type will be found in (7.126)–(7.166).

(48) $\int_o^\pi \log^2(2\sin\frac{1}{2}\theta)\log(2\cos\frac{1}{2}\theta)\,d\theta=\frac{1}{4}\pi\zeta(3)$

(49) $\mathrm{Ls}_5(\pi)=-19\pi^5/240$. See (7.113) for further values.

(50) $\mathrm{Ls}_n(\theta+2m\pi)=2m\,\mathrm{Ls}_n(\pi)+\mathrm{Ls}_n(\theta)$

(51) $\mathrm{Ls}_n^{(1)}(2m\pi)+\mathrm{Ls}_n^{(1)}(\theta)-\mathrm{Ls}_n^{(1)}(2m\pi-\theta)=2m\pi\,\mathrm{Ls}_{n-1}(\theta)$

(52) $\mathrm{Ls}_n^{(1)}(2m\pi)=2m^2\pi\,\mathrm{Ls}_{n-1}(\pi)$

(53) $\mathrm{Ls}_n^{(r)}(2m\pi)+(-1)^{r-1}\mathrm{Ls}_n^{(r)}(\theta)-\mathrm{Ls}_n^{(r)}(2m\pi-\theta)$

$$=\sum_{p=1}^{r}(-1)^{r-p}(2m\pi)^p\binom{r}{p}\mathrm{Ls}_{n-p}^{(r-p)}(\theta)$$

A.2.8. Summation of Series

(1) $\frac{v^2}{2}\frac{w}{1}-\frac{v^3}{3}\left(\frac{w}{1}-\frac{w^2}{2}\right)+\frac{v^4}{4}\left(\frac{w}{1}-\frac{w^2}{2}+\frac{w^3}{3}\right)-\cdots$

$$=\mathrm{Li}_2\left[\frac{(1+v)w}{1+w}\right]-\mathrm{Li}_2\left(\frac{w}{1+w}\right)+\log\left(\frac{1}{1+w}\right)\log(1+v),\ |v|<1,\ |w|<1$$

(2) $\frac{v^3}{3}\cdot\frac{w^2}{1}+\frac{v^5}{5}\left(\frac{w^2}{1}+\frac{w^4}{2}\right)+\frac{v^7}{7}\left(\frac{w^2}{1}+\frac{w^4}{2}+\frac{w^6}{3}\right)+\cdots$

$$=\mathrm{Li}_2\left[\frac{w(1+v)}{1+w}\right]+\mathrm{Li}_2\left[\frac{-w(1+v)}{1-w}\right]-\tfrac{1}{2}\mathrm{Li}_2\left[\frac{-w^2(1-v^2)}{1-w^2}\right]$$

$$+\tfrac{1}{2}\log(1-w^2)\log\left(\frac{1-v}{1+v}\right)+\tfrac{1}{4}\log^2\left(\frac{1+w}{1-w}\right)$$

(3) $\frac{x^2}{1^n2^m}-\frac{x^3}{2^n3^m}+\frac{x^4}{3^n4^m}-\cdots=-x\Big\{\mathrm{Li}_n(-x)-m\,\mathrm{Li}_{n-1}(-x)$

$$+\frac{m(m+1)}{2!}\mathrm{Li}_{n-2}(x)-\cdots+(-1)^{n-1}\frac{m(m+1)\cdots(m+n-2)}{(n-1)!}$$

$$\left[\mathrm{Li}_1(-x)+\frac{m+n-1}{n}\right]\Big\}+(-1)^n\Big[\mathrm{Li}_m(-x)+n\,\mathrm{Li}_{m-1}(-x)$$

$$+\frac{n(n+1)}{2!}\mathrm{Li}_{m-2}(-x)+\cdots+\frac{n(n+1)(n+2)\cdots(n+m-2)}{(m-1)!}\mathrm{Li}_1(-x)\Big],$$

where $\mathrm{Li}_1(-x)=-\log(1+x)$

(4) $\sum_2^\infty\frac{1}{n^2}\left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n-1}\right)=\sum_1^\infty\frac{1}{n^3}=\mathrm{Li}_3(1)$

(5) $\sum_2^\infty\frac{1}{n^3}\left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n-1}\right)=\frac{1}{4}\sum_1^\infty\frac{1}{n^4}=\frac{1}{4}\mathrm{Li}_4(1)=\pi^4/360$

(6) $\sum_{p=0}^\infty\frac{p+2}{2^p(p+1)^2}\left(1+\frac{1}{2}+\cdots+\frac{1}{p+1}\right)=2\zeta(3)+(1-\log 2)\pi^2/6$

(7) $\sum_{p=0}^\infty(-1)^p\frac{p+2}{(p+1)^2}\left(1+\frac{1}{2}\cdots+\frac{1}{p+1}\right)=\frac{5}{8}\zeta(3)+\frac{\pi^2}{12}-\frac{1}{2}\log^2 2$

(8) $\int_0^1 f(xu)\,\mathrm{Li}_q(u)\,du=\sum_{p=0}^\infty a_px^p\left[\sum_{k=0}^{q-2}(-1)^k\frac{\mathrm{Li}_{q-k}(1)}{(p+1)^{k+1}}\right]$

$$-\sum_{p=0}^\infty(-1)^q\frac{a_px^p}{(p+1)^q}\left(1+\frac{1}{2}+\cdots+\frac{1}{p+1}\right)$$

(9) If $x_n=\sum_{p=0}^{n-1}a^py_{n-p}/p!$, then $y_n=\sum_{p=o}^{n-1}(-1)^pa^px_{n-p}/p!$

(10) $\sum_1^\infty\frac{\cos(n\theta)}{(2\cos\theta)^nn^2}=\frac{\pi^2}{12}-\frac{1}{2}[\theta^2+\log^2(2\cos\theta)],\qquad -\frac{1}{3}\pi<\theta<\frac{1}{3}\pi$

(11) $\sum_1^\infty\frac{(2\cos\theta)^n\cos(n\theta)}{n^2}=(\theta-\frac{1}{2}\pi)^2,\qquad \frac{1}{3}\pi<\theta<\frac{2}{3}\pi$

(12) $\sum_1^\infty\frac{x^{n+1}}{(n+1)^2}\left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}\right)=\frac{1}{2}\log x\log^2(1-x)$

$$+\log(1-x)\,\mathrm{Li}_2(1-x)-\mathrm{Li}_3(1-x)+\mathrm{Li}_3(1)$$

(13) $\sum_1^\infty[\mathrm{Li}_{2n}(1)-1]=\frac{3}{4}$

(14) $\sum_1^\infty[\mathrm{Li}_{2n+1}(1)-1]=\frac{1}{4}$

(15) $\sum_1^\infty\frac{1}{2^{2n}}\mathrm{Li}_{2n}(1)=\frac{1}{2}$

(16) $\sum_1^\infty\frac{1}{2^{2n+1}}\mathrm{Li}_{2n+1}(1)=\log(2)-\frac{1}{2}$

See Section 8.3.5 for further details.

A.3. REFERENCE LIST OF INTEGRALS

A.3.1.

(1) $\int_o^x\frac{\log(1+ax^n)}{x}\,dx=-\frac{1}{n}\mathrm{Li}_2(-ax^n)$

(2) $\int_o^t\frac{\log(a+bt)}{c+et}\,dt=\frac{1}{e}\log\left(\frac{ae-bc}{e}\right)\log\left(\frac{c+et}{c}\right)-\frac{1}{e}\mathrm{Li}_2\left[\frac{b(c+et)}{bc-ae}\right]$

$$+\frac{1}{e}\mathrm{Li}_2\left(\frac{bc}{bc-ae}\right)$$

(3) $\displaystyle\int_o^t \frac{\log(a+bt)}{c+et}\,dt = \frac{1}{2e}\log^2\left[\frac{b}{e}(c+et)\right] - \frac{1}{2e}\log^2\left(\frac{bc}{e}\right)$

$\displaystyle + \frac{1}{e}\,\mathrm{Li}_2\left[\frac{bc-ae}{b(c+et)}\right] - \frac{1}{e}\,\mathrm{Li}_2\left(\frac{bc-ae}{bc}\right)$

(4) $\displaystyle V_{m,n} = \int \frac{x^m \log(x)}{(a+bx^2)^{n+1/2}}\,dx;$

then $\displaystyle \frac{x^m\log(x)}{(a+bx^2)^{n-1/2}} - \int \frac{x^{m-1}}{(a+bx^2)^{n-1/2}}\,dx$

$= mV_{m-1,n-1} - (2n-1)bV_{m+1,n} = maV_{m-1,n} - (2n-m-1)bV_{m+1,n}$
$= (2n-1)aV_{m-1,n} - (2n-m-1)V_{m-1,n-1}$

(5) $\displaystyle\int \frac{\log(y)}{\sqrt{1-y^2}}\,dy = -\tfrac{1}{2}\,\mathrm{Cl}_2(2\sin^{-1}y) - \sin^{-1}(y)\log 2$

(6) $\displaystyle\int \frac{\log(y)}{\sqrt{1+y^2}}\,dy = \tfrac{1}{2}\,\mathrm{Li}_2\left[\left(\sqrt{1+y^2}-y\right)^2\right] + \tfrac{1}{2}\log^2\left(\frac{\sqrt{1+y^2}+y}{2}\right)$

(7) $\displaystyle\int \frac{\log(y)}{\sqrt{y^2-1}}\,dy = \tfrac{1}{2}\,\mathrm{Li}_2\left[-\left(y-\sqrt{y^2-1}\right)^2\right] + \tfrac{1}{2}\log^2\left(\frac{y+\sqrt{y^2-1}}{2}\right)$

(8) $\displaystyle\int_o^x \frac{\log(1+x^2)}{1-x}\,dx = \tfrac{1}{4}\,\mathrm{Li}_2(-x^2) + \tfrac{1}{2}\,\mathrm{Li}_2\left(\frac{2x}{1+x^2}\right) - \mathrm{Li}_2(x)$

$\displaystyle + \tfrac{1}{4}\log^2(1+x^2) - \log(1-x)\log(1+x^2)$

A.3.2.

(1) $\displaystyle\int_o^t \frac{\log\left[\beta^2+(\alpha+t)^2\right]}{c+t}\,dt = \log[\beta^2+(\alpha-c)^2]\log\left(\frac{c+t}{c}\right)$

$\displaystyle -2\,\mathrm{Li}_2\left[\frac{c+t}{\sqrt{(c-\alpha)^2+\beta^2}},\theta\right] + 2\,\mathrm{Li}_2\left[\frac{c}{\sqrt{(c-\alpha)^2+\beta^2}},\theta\right]$, where

$\displaystyle \cos\theta = \frac{c-\alpha}{\sqrt{(c-\alpha)^2+\beta^2}}$

(2) $\displaystyle\int_o^x \frac{\log(A+Bx+Cx^2)}{a+bx}\,dx = \frac{1}{b}\left[\log(D/b^2)\log\frac{a+bx}{a}\right.$

$\displaystyle \left. -2\,\mathrm{Li}_2\left(\frac{a+bx}{\sqrt{D/C}},\theta\right) + 2\,\mathrm{Li}_2\left(\frac{a}{\sqrt{D/C}},\theta\right)\right],$

where $\cos\theta = (aC - \tfrac{1}{2}bB)/\sqrt{CD}$ and $D = Ab^2 - abB + Ca^2$

(3) $\displaystyle\int_o^t \frac{\delta\log(a+t)}{(\gamma+t)^2+\delta^2}\,dt = \tfrac{1}{2}\tan^{-1}\left(\frac{\delta t}{\delta^2+\gamma^2+\gamma t}\right)\log[\delta^2+(\alpha-\gamma)^2]$

$\displaystyle -\tfrac{1}{2}\,\mathrm{Cl}_2(2\theta+2\phi) + \tfrac{1}{2}\,\mathrm{Cl}_2(2\theta_0+2\phi) - \tfrac{1}{2}\,\mathrm{Cl}_2(\pi-2\theta) + \tfrac{1}{2}\,\mathrm{Cl}_2(\pi-2\theta_0),$

where $\tan\theta = (\gamma+t)/\delta$, $\tan\theta_0 = \gamma/\delta$, and $\tan\phi = (a-\gamma)/\delta$

(4) $\displaystyle\int_{-a}^t \frac{(\gamma+t)\log(a+t)}{(\gamma+t)^2+\delta^2}\,dt = \tfrac{1}{2}\log\left[\frac{(\gamma+t)^2+\delta^2}{(\gamma-a)^2+\delta^2}\right]\log(a+t)$

$\displaystyle + \mathrm{Li}_2\left(\frac{a+t}{M},\theta\right)$, where $M = \sqrt{\delta^2+(\gamma-a)^2}$, $\cos\theta = (a-\gamma)/M$

(5) $\displaystyle\int \frac{(\gamma+t)\log\left[\beta^2+(\gamma+t)^2\right]}{\delta^2+(\gamma+t)^2}\,dt = \tfrac{1}{2}\,\mathrm{Li}_2\left[\frac{\delta^2-\beta^2}{\delta^2+(\gamma+t)^2}\right]$

$\displaystyle + \tfrac{1}{4}\log\left[\frac{\delta^2+(\gamma+t)^2}{\delta^2}\right]\log[\delta^4+\delta^2(\gamma+t)^2]$

(6) $\displaystyle\int \frac{(\gamma+t)\log\left[\beta^2+(\alpha+t)^2\right]}{\delta^2+(\gamma+t)^2}\,dt = \log\left(\frac{1-\varepsilon\cos 2\phi}{1-\varepsilon}\right)\log\cos(\theta-\phi)$

$\displaystyle + \log\left[\frac{\cos\theta}{N(1-\varepsilon\cos 2\phi)}\right]\log(\cos\theta) + \tfrac{1}{2}\,\mathrm{Li}_2\left[-\frac{2\varepsilon}{1-\varepsilon}\cos^2(\theta-\phi)\right]$

$\displaystyle + 2\,\mathrm{Li}_2\left[\frac{\cos\theta}{\cos(\theta-\phi)}\sqrt{\frac{1-\varepsilon}{1-\varepsilon\cos 2\phi}},\psi\right] - 2\,\mathrm{Li}_2\left[\frac{\cos\theta}{\cos(\theta-\phi)},\phi\right],$

where $N = \tfrac{1}{2}[\delta^2+\beta^2+(\alpha-\gamma)^2]$, $\varepsilon = \sqrt{1-(\beta\delta/N)^2}$,

$\tan 2\phi = 2\delta(\alpha-\gamma)/[\beta^2+(\alpha-\gamma)^2-\delta^2]$, $\tan\theta = (\gamma+t)/\delta$,

$\displaystyle \cos\psi = \cos\phi\sqrt{\frac{1-\varepsilon}{1-\varepsilon\cos 2\phi}}$

(7) $\displaystyle\int \frac{\delta\log\left[\beta^2+(\alpha+t)^2\right]}{\delta^2+(\gamma+t)^2}\,dt = \theta\log 4N - \mathrm{Cl}_2(\pi-2\theta)$

$\displaystyle + (\theta-\phi)\log\left(\frac{1-\sqrt{1-\varepsilon^2}}{2}\right) - \eta\log\left(\frac{1-\sqrt{1-\varepsilon^2}}{\varepsilon}\right) + \tfrac{1}{2}\,\mathrm{Cl}_2(2\eta)$

$\displaystyle + \tfrac{1}{2}\,\mathrm{Cl}_2(4\theta-4\phi-2\eta) - \tfrac{1}{2}\,\mathrm{Cl}_2(4\theta-4\phi)$, where N, ε, ϕ, and θ are as in example (6) above and $\displaystyle\tan\eta = \frac{\varepsilon\sin(2\theta-2\phi)}{1+\varepsilon\cos(2\theta-2\phi)-\sqrt{1-\varepsilon^2}}$

A.3.3.

(1) $\displaystyle\int_o^\theta \log(\sinh\theta)\,d\theta = \tfrac{1}{2}\,\mathrm{Li}_2(e^{-2\theta}) - \theta\log 2 + \tfrac{1}{2}\theta^2 - \pi^2/12$

(2) $\int_o^\theta \log(\cosh\theta)\,d\theta = \frac{1}{2}\,\mathrm{Li}_2(-e^{-2\theta}) - \theta\log 2 + \frac{1}{2}\theta^2 + \pi^2/24$

(3) $\int_o^\theta \log(\tanh\theta)\,d\theta = \frac{1}{2}\,\mathrm{Li}_2(e^{-2\theta}) - \frac{1}{2}\,\mathrm{Li}_2(-e^{-2\theta}) - \pi^2/8$

(4) $\int_o^\theta \log(\sin\theta)\,d\theta = -\theta\log 2 - \frac{1}{2}\,\mathrm{Cl}_2(2\theta)$

(5) $\int_o^\theta \log(\cos\theta)\,d\theta = -\theta\log 2 + \frac{1}{2}\,\mathrm{Cl}_2(\pi - 2\theta)$

(6) $\int_o^\theta \log(\tan\theta)\,d\theta = -\frac{1}{2}\,\mathrm{Cl}_2(2\theta) - \frac{1}{2}\,\mathrm{Cl}_2(\pi - 2\theta)$

(7) $\int_o^\theta \frac{\log(\cos\theta)}{\sin 2\theta}\,d\theta = \frac{1}{8}\,\mathrm{Li}_2(-\tan^2\theta)$

(8) $\int_o^\theta \tan\theta\log(\tan\theta)\,d\theta = \frac{1}{4}\,\mathrm{Li}_2(-\tan^2\theta) - \log(\tan\theta)\log(\cos\theta)$

(9) $\int_o^{\frac{1}{2}\pi} \cot\theta\log(\cot\theta)\,d\theta = \frac{1}{4}\,\mathrm{Li}_2(-\cot^2\theta) - \log(\cot\theta)\log(\sin\theta)$

(10) $\int_o^\theta \tan\theta\log(\sin\theta)\,d\theta = \frac{1}{4}\,\mathrm{Li}_2(\cos^2\theta) - \pi^2/24$

(11) $\int_o^\theta \cot\theta\log(\cos\theta)\,d\theta = -\frac{1}{4}\,\mathrm{Li}_2(\sin^2\theta)$

(12) $\int_o^\theta \tan\theta\log(\cos\theta)\,d\theta = -\frac{1}{2}\log^2(\cos\theta)$

(13) $\int_o^\theta \frac{\theta\,d\theta}{\sin\theta} = \mathrm{Cl}_2(\theta) + \mathrm{Cl}_2(\pi-\theta) + \theta\log(\tan\frac{1}{2}\theta)$

(14) $\int_o^\theta \theta\cot(\theta)\,d\theta = \frac{1}{2}\,\mathrm{Cl}_2(2\theta) + \theta\log(2\sin\theta)$

(15) $\int_o^\theta \theta\tan(\theta)\,d\theta = \frac{1}{2}\,\mathrm{Cl}_2(\pi - 2\theta) - \theta\log(2\cos\theta)$

(16) $\int_o^\theta \theta^2\sec^2(\theta)\,d\theta = \theta^2\tan\theta + 2\theta\log(2\cos\theta) - \mathrm{Cl}_2(\pi - 2\theta)$

(17) $\int_o^\theta \theta^2\,\mathrm{cosec}^2(\theta)\,d\theta = \mathrm{Cl}_2(2\theta) + 2\theta\log(2\sin\theta) - \theta^2\cot\theta$

(18) $\int_o^\theta \theta^2\cos(\theta)\,\mathrm{cosec}^2(\theta)\,d\theta = 2\,\mathrm{Cl}_2(\theta) + 2\,\mathrm{Cl}_2(\pi-\theta) + 2\theta\log(\tan\frac{1}{2}\theta)$
$-\theta^2\,\mathrm{cosec}(\theta)$

(19) $\int_o^x \frac{\sin^{-1}(x)}{x}\,dx = \frac{1}{2}\,\mathrm{Cl}_2(2\sin^{-1}x) + \sin^{-1}(x)\log(2x)$

(20) $\int_o^\theta \tan^{-1}\left(\frac{a\sin\theta}{1-a\cos\theta}\right)d\theta = \mathrm{Li}_2(a) - \mathrm{Li}_2(a,\theta)$

(21) $\int_o^\theta \tan^{-1}\left(\frac{\sin\phi + a\sin\theta}{\cos\phi - a\cos\theta}\right)d\theta = \theta\phi + \mathrm{Li}_2(a,\phi) - \mathrm{Li}_2(a,\theta+\phi)$

(22) $\int_o^\theta \frac{\theta\,d\theta}{1+a^2-2a\cos(\theta+\phi)} = \frac{2}{1-a^2}\left[\frac{\theta^2}{4} - \theta\phi + \theta\tan^{-1}\left(\frac{\sin\phi + a\sin\theta}{\cos\phi - a\cos\theta}\right)\right.$
$\left. + \mathrm{Li}_2(a,\theta+\phi) - \mathrm{Li}_2(a,\phi)\right]$

(23) $\int_o^\phi \cot(\theta+\phi)\log(\sin\phi)\,d\phi = \pi^2/6 + \frac{1}{2}\,\phi(\phi + 2\theta - 2\pi)$
$+ \log(\sin\theta)\log\left[\frac{\sin(\theta+\phi)}{\sin\theta}\right] - \mathrm{Li}_2\left[\frac{\sin(\theta+\phi)}{\sin\theta},\phi\right]$

(24) $\int_o^\phi \cot(\theta-\phi)\log(\sin\phi)\,d\phi = -\pi^2/6 + \frac{1}{2}\,\phi(2\theta - \phi)$
$-\log(\sin\theta)\log\left[\frac{\sin(\theta-\phi)}{\sin\theta}\right] + \mathrm{Li}_2\left[\frac{\sin(\theta-\phi)}{\sin\theta},\phi\right]$

(25) $\int_o^\phi \frac{\sin 2\theta}{\cos 2\phi - \cos 2\theta}\log(\sin\phi)\,d\phi = \mathrm{Li}_2\left[\frac{\sin(\theta-\phi)}{\sin\theta},\phi\right]$
$-\mathrm{Li}_2\left[\frac{\sin(\theta+\phi)}{\sin\theta},\phi\right] + \log(\sin\theta)\log\left[\frac{\sin(\theta+\phi)}{\sin(\theta-\phi)}\right] + \phi(2\theta - \pi)$

(26) $\int_o^\phi \frac{\sin 2\phi}{\cos 2\phi - \cos 2\theta}\log(\sin\phi)\,d\phi = \mathrm{Li}_2\left[\frac{\sin(\theta-\phi)}{\sin\theta},\phi\right]$
$+ \mathrm{Li}_2\left[\frac{\sin(\theta+\phi)}{\sin\theta},\phi\right] + \log(\sin\theta)\log\left[\frac{1-\cos 2\theta}{\cos 2\phi - \cos 2\theta}\right]$
$+ \phi(\pi - \phi) - \pi^2/3$

(27) $\int_o^{\tan^{-1}\left(\frac{\sin\theta}{\cos\theta - a}\right)} \sin^{-1}(a\sin\phi)\,d\phi = \frac{1}{2}\left[\tan^{-1}\left(\frac{a\sin\theta}{1-a\cos\theta}\right)\right]^2$
$+ \mathrm{Li}_2(a) - \mathrm{Li}_2(a,\theta)$

(28) $\int_o^\theta \tan^{-1}(b\tan\theta)\,d\theta = \frac{1}{2}\,\mathrm{Li}_2[(1+b)\sin\theta, \frac{1}{2}\pi - \theta]$
$-\frac{1}{2}\,\mathrm{Li}_2[(1-b)\sin\theta, \frac{1}{2}\pi - \theta]$

(29) $\int_{-\phi}^\theta \tan^{-1}(a + b\tan\theta)\,d\theta = -(\theta+\phi)\tan^{-1}(B^{-1}\tan\phi)$
$+ \frac{1}{2}\,\mathrm{Li}_2[(1+B)\sin(\theta+\phi), \frac{1}{2}\pi - \theta - \phi] - \frac{1}{2}\,\mathrm{Li}_2[(1-B)\sin(\theta+\phi), \frac{1}{2}\pi - \theta$
$-\phi]$, where $B = \tan(\frac{1}{2}\beta), \sin\beta = 2b/(1+a^2+b^2), \tan 2\phi$
$= -2ab/(1+a^2-b^2)$

(30) $\int_o^\theta \frac{\theta\,d\theta}{\sin(\phi) + \sin(2\theta+\phi)} = \frac{1}{2}\sec\phi\left\{\theta\log\left[\frac{\sin(\theta+\phi)}{\cos\theta}\right] + \frac{1}{2}\,\mathrm{Cl}_2(2\theta + 2\phi)\right.$
$\left. - \frac{1}{2}\,\mathrm{Cl}_2(2\phi) + \frac{1}{2}\,\mathrm{Cl}_2(\pi - 2\theta)\right\}$

(31) $\int \frac{e\tan^{-1}(a+bx)}{c+ex}\,dx = \theta\log\left[\frac{\sin(\theta+\phi)}{\cos\theta}\right]$
$+ \frac{1}{2}\,\mathrm{Cl}_2(2\phi + 2\theta) + \frac{1}{2}\,\mathrm{Cl}_2(\pi - 2\theta)$,
where $\tan\phi = (bc - ae)/e$, $\tan\theta = a + bx$

(32) To integrate $\int \frac{f\tan^{-1}(a+bx)}{(c+ex)^2 + f^2}\,dx$ and $\int \frac{(c+ex)\tan^{-1}(a+bx)}{(c+ex)^2+f^2}\,dx$
put $c + ex = f\tan\theta$ and use previous results

(33) $\int_\phi^\theta \log(\sin^2\theta-\sin^2\phi)\,d\theta=\int_\phi^\theta \log(\cos^2\phi-\cos^2\theta)\,d\theta$
$=\frac{1}{2}\mathrm{Cl}_2(4\phi)-\frac{1}{2}\mathrm{Cl}_2(2\theta+2\phi)-\frac{1}{2}\mathrm{Cl}_2(2\theta-2\phi)-2(\theta-\phi)\log 2$

(34) $\int_\theta^\phi \log(\sin^2\phi-\sin^2\theta)\,d\theta=\int_\theta^\phi \log(\cos^2\theta-\cos^2\phi)\,d\theta$
$=\frac{1}{2}\mathrm{Cl}_2(2\theta+2\phi)-\frac{1}{2}\mathrm{Cl}_2(4\phi)-\frac{1}{2}\mathrm{Cl}_2(2\phi-2\theta)+2(\theta-\phi)\log 2$

(35) $\int_\phi^\theta \log(\cos\phi-\cos\theta)\,d\theta=\mathrm{Cl}_2(2\phi)-\mathrm{Cl}_2(\theta+\phi)-\mathrm{Cl}_2(\theta-\phi)$
$-(\theta-\phi)\log 2$

(36) $\int_o^\theta \log(1+\sec\phi\cos\theta)\,d\theta=-\theta\log(2\cos\phi)+\mathrm{Cl}_2(\pi+\phi-\theta)$
$+\mathrm{Cl}_2(\pi-\phi-\theta)$

(37) $\int_o^\theta \log(\tan\theta+\tan\phi)\,d\theta=-\theta\log(\cos\phi)-\frac{1}{2}\mathrm{Cl}_2(2\theta+2\phi)$
$+\frac{1}{2}\mathrm{Cl}_2(2\phi)-\frac{1}{2}\mathrm{Cl}_2(\pi-2\theta)$

(38) $\int_o^\theta \log(1+\sin\phi\cos\theta)\,d\theta=\theta\log(\sin^2\frac{1}{2}\phi)-\eta\log(\tan^2\frac{1}{2}\phi)-\mathrm{Cl}_2(2\theta)$
$+\mathrm{Cl}_2(2\theta-2\eta)+\mathrm{Cl}_2(2\eta)$, where $\tan\eta=\sin\theta/(\tan\frac{1}{2}\phi+\cos\theta)$

(39) $\int_o^\theta \log(1-2r\cos\theta+r^2)\,d\theta=\mathrm{Cl}_2(2\theta+2\omega)-\mathrm{Cl}_2(2\theta)-\mathrm{Cl}_2(2\omega)$
$-2\omega\log(r)$, where $\tan\omega=r\sin\theta/(1-r\cos\theta)$

(40) $\int \log[a^2+(b\tan\theta+c)^2]\,d\theta$ can be deduced by putting $b\tan\theta=t$ and using previous results

(41) $\int \log(1-2r\cos\alpha\sin\theta+r^2\sin^2\theta)\,d\theta$ can be deduced by putting $\tan\frac{1}{2}\theta=t$; see (8.70)

A.3.4.

(1) $\int_o^1 x^{\alpha-1}\mathrm{Li}_2(x)\,dx=\pi^2/6\alpha-[\Psi(1+\alpha)+\gamma]/\alpha^2$

(2) $\int_o^1 x^n\mathrm{Li}_2(x)\,dx=\dfrac{\pi^2}{6(n+1)}-\dfrac{1}{(n+1)^2}\left(\dfrac{1}{1}+\dfrac{1}{2}+\cdots+\dfrac{1}{n+1}\right)$

(3) $\int_o^1 \dfrac{\mathrm{Li}_2(t)\,dt}{(1-xt)^2}=\dfrac{\pi^2}{6(1-x)}-\dfrac{\mathrm{Li}_2(x)}{x}-\dfrac{\log^2(1-x)}{2x},\qquad x<1$

(4) $\int_o^{\pi/2}\mathrm{Li}_2(-y^2\tan^2\theta)\,d\theta=2\pi\mathrm{Li}_2(-y),\qquad y\geqq 0$

(5) $\int_o^1 \dfrac{\log(1-2x\cos\theta+x^2)}{x}\,dx=\pi^2/6-\frac{1}{2}(\pi-\theta)^2,\qquad 0\leqq\theta\leqq 2\pi$

(6) $\int_o^{\pi/2}\sin^{-1}(a\sin\theta)\,d\theta=\chi_2(a),\qquad 0\leqq a\leqq 1$

(7) $\int_o^\infty \dfrac{\tan^{-1}(ax)}{1+x^2}\,dx=\pi^2/6-\frac{1}{4}\log^2(1+a)-\frac{1}{2}\mathrm{Li}_2\left(\dfrac{1}{1+a}\right)-\frac{1}{2}\mathrm{Li}_2(1-a)$
$=\dfrac{\pi^2}{8}-\chi_2\left(\dfrac{1-a}{1+a}\right)$

(8) $\int_o^{\pi/2}\tan^{-1}(A\operatorname{cosec}x)\,dx=\pi^2/4-2\chi_2[(1+A^2)^{1/2}-A]$

(9) $\int_o^{\pi/2}\dfrac{x^2\,dx}{1-P\cos 2x}=\dfrac{1+p^2}{1-p^2}\left[\dfrac{\pi^3}{24}+\dfrac{\pi}{2}\mathrm{Li}_2(-p)\right],\qquad p^2<1,\ P=\dfrac{2p}{1+p^2}$

(10) $\int_o^\pi \dfrac{x^2}{1-Q\cos^2 x}\,dx=\dfrac{1+q}{1-q}\left[\dfrac{\pi^3}{3}+\pi\mathrm{Li}_2(q)\right],\qquad q<1,\ Q=\dfrac{4q}{(1+q)^2}$

(11) $\int_o^\pi \dfrac{x^2}{1-R\cos 2x}\,dx=\dfrac{1+r^2}{1-r^2}\left[\dfrac{\pi^3}{3}+\pi\mathrm{Li}_2(r)\right],\qquad r<1,\ R=\dfrac{2r}{1+r^2}$

(12) $\int_o^\pi \dfrac{x^2\cos x}{1-R\cos 2x}\,dx=-2\pi\dfrac{1+r^2}{1-r}\dfrac{\chi_2(r^{1/2})}{r^{1/2}},\qquad r<1,\ R=\dfrac{2r}{1+r^2}$

(13) $\int_o^\pi x\tan^{-1}\left[\dfrac{2p}{1-p^2}\sin x\right]dx=2\pi\chi_2(p),\qquad p^2<1$

(14) $(1/\pi)\int_o^\pi \sec x\,\mathrm{Li}_2(\sin\theta\cos x)\,dx=\mathrm{Cl}_2(2\theta)+\theta\log(2\sin\theta)$

(15) $\int_o^\pi\int_{\phi/2}^{(\pi+\phi)/2}\log(1-B^2\cos^2\theta)\,d\theta\,d\phi=4\chi_3(p)-\frac{1}{2}\pi^2\log C$
with $p=B/D,\quad C=4/D,\quad D=2-B+2(1-B)^{1/2}$

(16) $\int_o^{\sin^{-1}s}\sin^{-1}\left(\dfrac{\sin\psi}{s}\right)\mathrm{Cl}_2(2\psi)\,d\psi=\dfrac{\pi}{4}[\mathrm{Cl}_3(\pi)-\mathrm{Cl}_3(2\sin^{-1}s)$
$+\log s\,\mathrm{Li}_2(s^2)-\mathrm{Li}_3(s^2)],\qquad s>0$

(17) $\int_o^1 \log(x)\log^2(1-x)\,dx/x=-\pi^4/180$

A.3.5.

(1) $\int_o^x \log(1-y)\log(1-cy)\,dy/y=\mathrm{Li}_3\left(\dfrac{1-xc}{1-x}\right)-\mathrm{Li}_3\left(\dfrac{1-cx}{c(1-x)}\right)$
$-\mathrm{Li}_3(1-x)-\mathrm{Li}_3(1-cx)+\mathrm{Li}_3(1/c)+\mathrm{Li}_3(1)+\log(1-cx)[\mathrm{Li}_2(1/c)$
$-\mathrm{Li}_2(x)]+\log(1-x)[\mathrm{Li}_2(1-cx)-\mathrm{Li}_2(1/c)+\pi^2/6]$
$+\frac{1}{2}\log(c)\log^2(1-x)$

(2) $\int \dfrac{\log(a+bx)\log(c+ex)}{f+gx}\,g\,dx=\log\left(\dfrac{g}{cg-ef}\right)\mathrm{Li}_2(y)$
$+\log\left(\dfrac{g}{ag-bf}\right)\mathrm{Li}_2(Ky)+\log\left(\dfrac{g}{cg-ef}\right)\log\left(\dfrac{g}{ag-bf}\right)\log y$
$+\int \log(1-y)\log(1-Ky)\,dy/y$, where $K=\dfrac{e}{b}\cdot\dfrac{ag-bf}{cg-ef},\ y=\dfrac{b(f+gx)}{bf-ag}$,
and example (1) above is used for the last integral; this result holds provided $ag-bf\neq 0$ and $cg-ef\neq 0$ [see examples (3)–(5) below]

(3) In example (2) if $ag-bf\neq 0$ but $cg-ef=0$, then the integral equals

$$\log\left[\frac{bg}{e(bf-ag)}\right]\mathrm{Li}_2(y)+\log\left[\frac{bg}{e(bf-ag)}\right]\log\left(\frac{g}{ag-bf}\right)\log y$$
$$-\tfrac{1}{2}\log\left(\frac{g}{ag-bf}\right)\log^2(y)+\int\log(y)\log(1-y)\,dy/y$$

(4) If $ag-bf=0$ but $cg-ef\neq 0$, then the integral equals

$$\log\left[\frac{eg}{b(ef-cg)}\right]\mathrm{Li}_2(z)+\log\left[\frac{eg}{b(ef-cg)}\right]\log\left(\frac{g}{cg-ef}\right)\log z$$
$$-\tfrac{1}{2}\log\left(\frac{g}{cg-ef}\right)\log^2(z)+\int\log(z)\log(1-z)\,dz/z,\text{ where}$$
$$z=\frac{e(f+gx)}{ef-cg}$$

(5) If $ag-bf=0$ and $cg-ef=0$, then the integral equals $\frac{1}{2}\log(e/b)\log^2(v)+\frac{1}{3}\log^3(v)$, where $v=b(f+gx)/g$

(6) $\displaystyle\int_o^x \log^2(1-x)\,dx/x=\log(x)\log^2(1-x)+2\log(1-x)\mathrm{Li}_2(1-x)$
$-2\mathrm{Li}_3(1-x)+2\mathrm{Li}_3(1)$

(7) $\displaystyle\int_o^x \log^2(1+x)\,dx/x=\log(x)\log^2(1+x)-\tfrac{2}{3}\log^3(1+x)$
$\displaystyle-2\log(1+x)\mathrm{Li}_2\left(\frac{1}{1+x}\right)-2\mathrm{Li}_3\left(\frac{1}{1+x}\right)+2\mathrm{Li}_3(1)$

(8) $\displaystyle\int_o^x \log(x)\log(1-x)\,dx/x=\mathrm{Li}_3(x)-\log(x)\mathrm{Li}_2(x)$

(9) $\displaystyle\int_1^x \log(x)\log(x-1)\,dx/x=\tfrac{1}{3}\log^3(x)+\log(x)\mathrm{Li}_2(1/x)-2\mathrm{Li}_3(1/x)+$
$2\mathrm{Li}_3(1)$

Further results of this character are given in Section 6.4.4.

(10) $\displaystyle\int_o^x \log^2(x)\log(1-x)\,dx/x=-2\mathrm{Li}_4(x)+2\log(x)\mathrm{Li}_3(x)$
$-\log^2(x)\mathrm{Li}_2(x)$

(11) $\displaystyle\int_o^x \log^3(x)\,dx/(1-x)=-6\mathrm{Li}_4(x)+6\log(x)\mathrm{Li}_3(x)-3\log^2(x)\mathrm{Li}_2(x)$
$-\log^3(x)\log(1-x)$

(12) $\displaystyle\int_o^x \log^2(x)\log(1-x)\,dx/(1-x)=2\Big[\mathrm{Li}_4(1-x)$
$\displaystyle-\mathrm{Li}_4(x)-\mathrm{Li}_4\left(\frac{-x}{1-x}\right)-\mathrm{Li}_4(1)\Big]+2\ \log(1-x)\mathrm{Li}_3(x)-\log(x)\mathrm{Li}_3(1-x)$
$\displaystyle+\mathrm{Li}_3(1)\log\left(\frac{x}{1-x}\right)\Big]+2\log(x)\log(1-x)\mathrm{Li}_2(1-x)$
$+\frac{1}{12}\log^2(1-x)\left[6\log^2(x)+4\log(x)\log(1-x)-\log^2(1-x)-2\pi^2\right]$

For further results of this type see Section 7.6.1. Equations (7.48) and (7.49) give similar results for higher powers of the logarithms.

The limiting value $x=1$ in Equation (12) gives

(13) $\displaystyle\int_o^1 \log(x)\log^2(1-x)\,dx/x=\int_o^1 \log^2(x)\log(1-x)\,dx/(1-x)$
$=-\pi^4/180$

A.3.6.

(1) $\displaystyle\int_o^\omega \log(2\sin\omega)\log[2\sin(\omega+\theta)]\,d\omega=-\tfrac{1}{2}\mathrm{Ls}_3(2\omega)+\tfrac{1}{2}\log(r)\mathrm{Ls}_2(2\omega)$

(2) $\displaystyle\int_o^\theta \log^2[2\sin(\theta+\omega)]\,d\theta=-\omega\log^2(r)-\log(r)\mathrm{Ls}_2(2\omega)+\tfrac{1}{2}\mathrm{Ls}_3(2\omega)$
$-\tfrac{1}{2}\mathrm{Ls}_3(2\theta+2\omega)$

(3) $\displaystyle\int_o^\omega \log^2[2\sin(\theta+\omega)]\,d\omega=-\omega\log^2(r)-\log(r)\mathrm{Ls}_2(2\omega)+\tfrac{1}{2}\mathrm{Ls}_3(2\omega)$

(4) $\displaystyle\int_o^\theta \log(2\sin\omega)\log[2\sin(\omega+\theta)]\,d\theta=-\tfrac{1}{2}\log(r)[\mathrm{Ls}_2(2\omega)+\mathrm{Ls}_2(2\theta+2\omega)]$
$+\tfrac{1}{2}\mathrm{Ls}_3(2\omega)-\tfrac{1}{2}\mathrm{Ls}_3(2\theta+2\omega)$

(5) $\displaystyle\int_o^\theta \log^2(2\sin\omega)\,d\theta=\tfrac{1}{2}\mathrm{Ls}_3(2\omega)-\tfrac{1}{2}\mathrm{Ls}_3(2\omega+2\theta)+(\omega+\theta)\log^2(r)$
$-\log(r)\mathrm{Ls}_2(2\omega+2\theta)$.

In the above five formulas ω and θ are related by the equation $\dfrac{\sin(\omega)}{\sin(\omega+\theta)}=r$, where r is a constant

A.3.7.

(1) $\displaystyle\int_0^\infty \frac{1-e^{-at}}{t}\mathrm{Ci}(t)\,dt=\tfrac{1}{4}\mathrm{Li}_2(-a^2)$

(2) $\displaystyle\int_0^\infty \frac{1-e^{-at}}{t}\mathrm{si}(t)\,dt=\mathrm{Ti}_2(a)$

(3) $\displaystyle\int_0^\infty \frac{1-e^{-at}}{t}\mathrm{Ei}(-t)\,dt=\mathrm{Li}_2(-a)$

(4) $\displaystyle\int_0^\infty \frac{e^{-at}}{t}\mathrm{Si}(t)\,dt=\mathrm{Ti}_2(1/a)$

(5) $\displaystyle\lim_{\varepsilon\to 0}-\int_\varepsilon^\infty \frac{e^{-at}}{t}\mathrm{Ei}(-t)\,dt=\mathrm{Li}_2(-a)+\pi^2/12+\tfrac{1}{2}(\gamma+\log\varepsilon)^2$

(6) $\displaystyle\int_0^\infty e^{-px}\mathrm{Li}_2(-p^2)\,dp=-(2/x)[\mathrm{Ci}^2(x)+\mathrm{si}^2(x)]$

(7) $\displaystyle\int_0^\infty e^{-px}\mathrm{Li}_2(-p)\,dp=(1/x)\int_x^\infty \frac{e^x}{x}\mathrm{Ei}(-x)\,dx$

For other results of a similar character see Section 1.9.5.

A.3.8.

(1) $\mathrm{Li}_\nu(z)+\mathrm{Li}_{-\nu}(z)=2^{1-\nu}\mathrm{Li}_\nu(z^2)$

(2) $\mathrm{Li}_\nu(z)=\dfrac{1}{\Gamma(\nu)}\displaystyle\int_0^\infty \frac{zt^{\nu-1}\,dt}{e^t-z}, \qquad \mathrm{Re}\,\nu>0$

(3) $\mathrm{Li}_\nu(z)+e^{i\pi\nu}\mathrm{Li}_\nu(1/z)=\dfrac{e^{i\pi\nu/2}(2\pi)^\nu}{\Gamma(\nu)}\,\zeta\left(1-\nu,\dfrac{\log z}{2\pi i}\right), \qquad \mathrm{Re}\,\nu<0$

(4) $\displaystyle\lim_{z\to 1}(1-z)^{1-\nu}\mathrm{Li}_\nu(z)=\Gamma(1-\nu), \qquad 0<\mathrm{Re}\,\nu<1$

A.4. TABULATED VALUES

A.4.1. Several of the earlier writers spent a good deal of effort in preparing tables of the various functions we have studied. The convergence of series expansions was sometimes very slow and much work was put into developing alternative expansions for particular ranges of the variables. Considerable use was made of the various functional equations, and in fact these were often judged mainly on their merits as aids to computation. Among the earliest and perhaps most remarkable of these works is Clausen's sixteen-decimal tabulation* of the log-sine integral at 1° intervals. The Fourier sine series would appear to be quite unsuitable for computation to this order of accuracy, at least not without some extensive manipulation, and we are, perhaps, in some sympathy with Newman, who admits rather dolefully that he did not know how it was done.

Spence* tabulated, at unit intervals, and to nine decimal figures, his second- and third-order logarithmic functions, together with the function

$$\overset{1}{C}(x)\equiv\tan^{-1}(x)/x.$$

A small table of the same nature for the dilogarithm was also published by Hill,* and an eleven-decimal tabulation of $\Lambda_2(x)$ at unit intervals was given by Kummer* in 1840.

The first comprehensive table for the dilogarithm was given by Newman* from values computed by J. C. Adams. More recently, Powell* has given a tabulation of the related $\int_x^1[\log x/(1-x)]dx$ to seven places of decimals with x from 0 to 6 in 0.01 steps; and a short note by Fletcher* corrects some values in these tables. A nine-decimal tabulation of $\int_0^x -(\log|1-y|/y)\,dy$ with $x=-1.00(0.01)1.00$ has been given by Mitchell* in a short paper reviewing some of the uses and properties of the dilogarithm.

*The locations of these tables are given in the references in the bibliography, sections B.2 and B.3.

In connection with an electrical problem the function

$$\frac{1}{\pi}\int_0^x \log\frac{|1+t|}{|1-t|}\,\frac{dt}{t},$$

which is closely related to Legendre's χ_2-function, has been tabulated by Corrington* to five decimal places.

A more recent tabulation (1956) of this same function by Thomas to seven-figure accuracy is given in Bell System monograph No. 2550. The tabulation interval for the independent variable has been chosen to permit linear interpolation to one unit in the last decimal place.

A number of special functions occuring in physical problems have also been tabulated and are discussed in Section 1.12. One might mention here a short table for polylogarithms of order $-\frac{1}{2}$, $\frac{1}{2}$ and $\frac{3}{2}$ given by Truesdell on page 150 of *Annals of Mathematics*, 1945 (University of Michigan).

A tabulation of $\mathrm{Cl}_2(\theta)$ to six decimal places at 1/6° intervals is given by Ashour and Sabri in *Mathematical Tables and other Aids to Computation*, April, 1956, pp. 57–65. This supplements Clausen's table, whose tabulation interval of 1° has been found too coarse for recent applications.

Modern computational methods have somewhat eclipsed the results of the earlier workers, but have made available a range of tables which would not have been likely to have been prepared in any other way. In 1954 the staff of the computation department of the Mathematisch Centrum, Amsterdam, under the leadership of Prof. Dr. Ir. A. van Wijngaarden produced a report on the numerical values of the polylogarithms for the first twelve orders, to ten decimal places. For real argument the variable ranges from $-1(0.01)$ to $+1$, and for imaginary argument from 0(0.01) to 1. The complex argument (modulus unity) is taken in the form $e^{i\pi\alpha/2}$ with $\alpha=0(0.01)2$.

A notable absentee from this list is the function $\mathrm{Li}_n(r,\theta)$ with $\mathrm{Li}_2(r,\theta)$, which occurs in a number of physical problems, as the most important. In view of the prominence given to this function throughout the book it was felt desirable that this particular gap should be filled, and a six-decimal tabulation with $r=0(0.01)1$ and $\theta=0(5°)180°$ has been prepared by the Scientific Computation Service and is presented in Table V. For the other functions the reader cannot do better than refer to Report R24 of the Mathematisch Centrum mentioned above. Thanks to the kind permission of Dr. van Wijngaarden it has, however, been possible to include a number of extracts from these tables, which it is hoped may be of some use. The first four functions are given, with the values rounded off to the fifth decimal place, for positive values of the argument. By the use of formulas

*See footnote to p. 312

already given the range of the tables can be extended: the more important of these formulas are given with the tables.

The polylogarithms of real argument are given in Table 1, the inverse tangent integrals in Table 2, and the Clausen and associated Clausen functions in Tables 3 and 4.

Much recent work has been done toward providing efficient algorithms for the numerical computation of polylogarithms. Computational methods as such are not within the scope of this book, but in Section B.3 the reader will find some references to recent work in this field. There are also some recent calculations by Fornberg and Kölbig on the location of the complex zeroes of the polylogarithms.

A.4.2. The relation $\mathrm{Li}_n(-x)=2^{1-n}\mathrm{Li}_n(x^2)-\mathrm{Li}_n(x)$ extends the tables to negative values as far as $x=1$. For larger values the inversion relation should be used. For the first four functions this equation gives, for positive x,

$$\mathrm{Li}_2(-x)=-\pi^2/6-\tfrac{1}{2}\log^2(x)-\mathrm{Li}_2(-1/x),$$

$$\mathrm{Li}_3(-x)=-\pi^2/6\log(x)-\tfrac{1}{6}\log^3(x)+\mathrm{Li}_3(-1/x),$$

$$\mathrm{Li}_4(-x)=-7\pi^4/360-\frac{\pi^2}{12}\log^2(x)-\tfrac{1}{24}\log^4(x)-\mathrm{Li}_4(-1/x),$$

$$\mathrm{Li}_5(-x)=-\frac{7\pi^4}{360}\log(x)-\frac{\pi^2}{36}\log^3(x)-\tfrac{1}{120}\log^5(x)+\mathrm{Li}_5(-1/x).$$

The values of the constants may be obtained via the tables by using the value $x=1$.

For $n=2$ the value of $\mathrm{Li}_2(-x)$ may be obtained from the values of $\mathrm{Li}_2(x,\theta)$ in Table 5 with $\theta=180°$.

APPENDIX

REFERENCE DATA AND TABLES