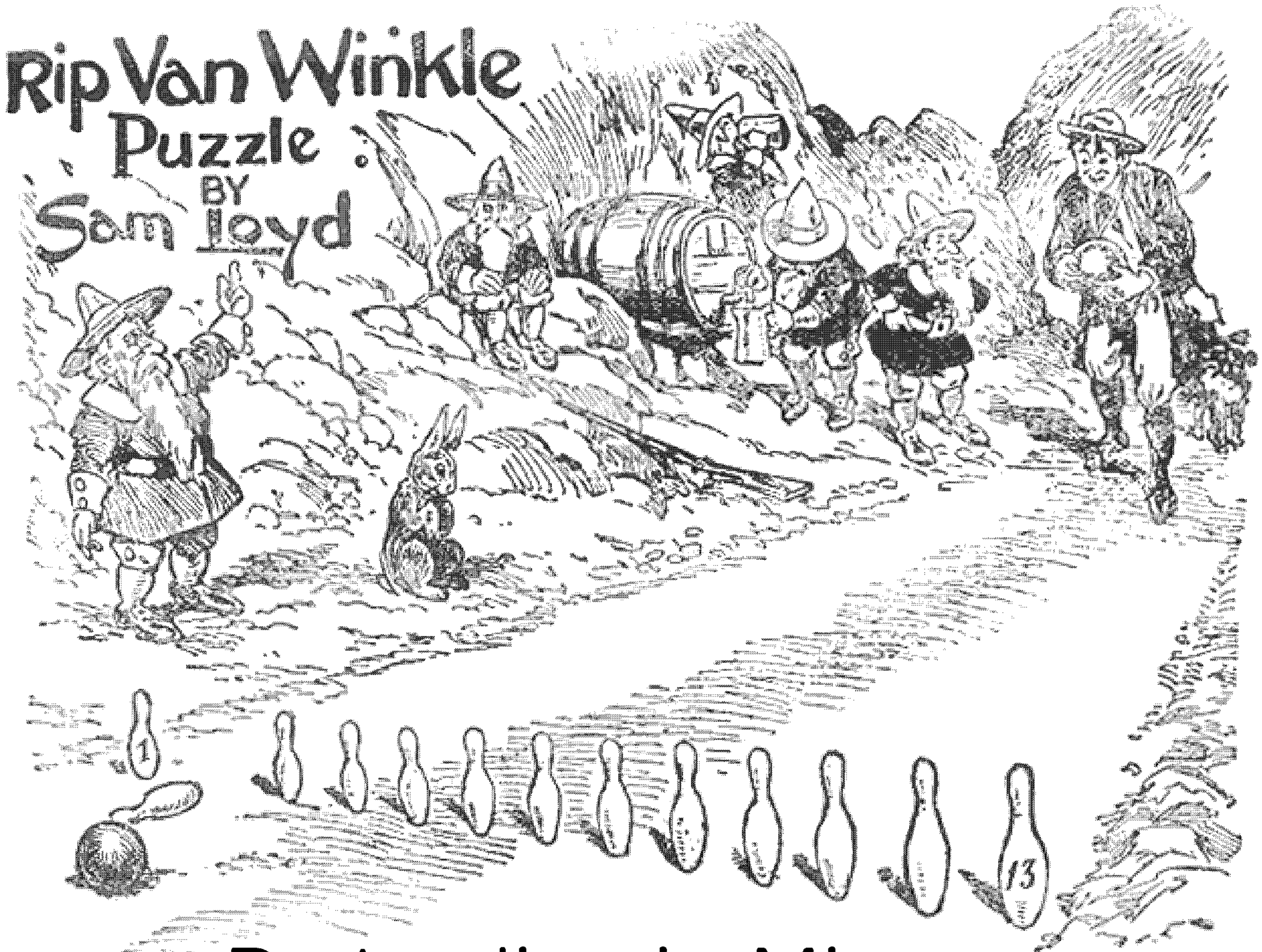

Pretending in Misere Combinatorial Games

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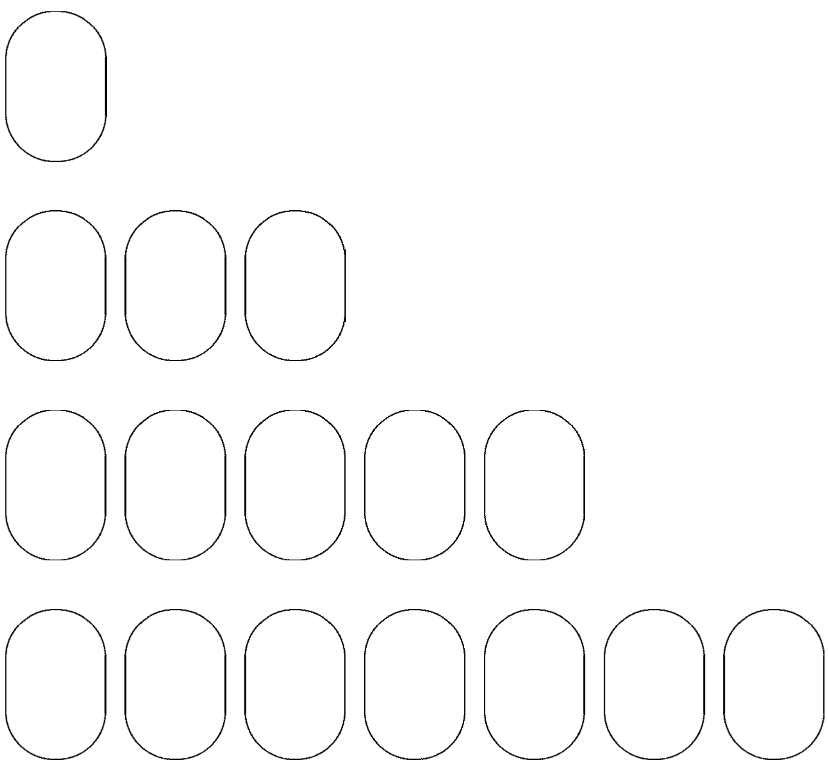
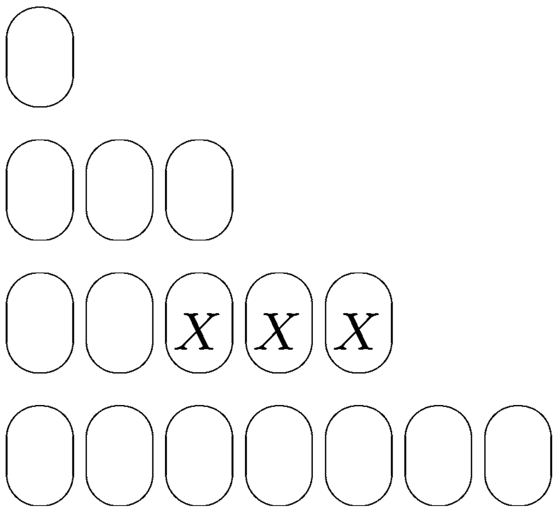
April 1, 2004

The game of Nim



How to win at normal play Nim



$(001)_2$

$(011)_2$

$(010)_2$

$(111)_2$

$\overline{}$

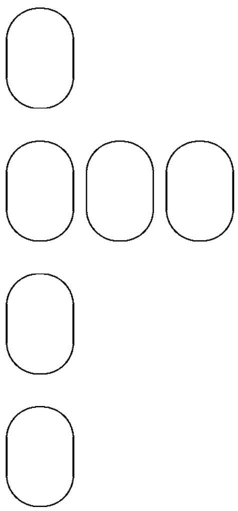
$(111)_2 =$ nim sum

1. Let the other person go first.

2. Make moves yielding *nim sum* zero.

3. *Nim sum*: add without carrying.

How to win at misère play Nim



$(001)_2$

$(011)_2$

$(001)_2$

$(001)_2$

$---$

$(010)_2 = \text{nim sum}$

1. Use the same strategy as before, EXCEPT...

2. ...if your move would leave heaps of size one only. In that case, leave either one more or one fewer than the normal play strategy recommends.

Impartial Combinatorial Games

1. Two players.
2. The **rules** specify **moves** to **options**.
3. Play must end after a finite number of moves.
4. **Normal play**: last player to move *wins*.
5. **Misere play**: last player to move *loses*.
6. No ties (ie, no "draws").
7. **Complete information**, and no **chance**.
8. **Impartiality**: From any position exactly the same moves are available to either player.

Impartial games have only two **outcome classes**:

1. P-position: (Previous player wins in best play)
2. N-position: (Next player wins in best play)

A **sum** of games $G + H$ is played **disjunctively**, ie, a player must select a summand and move in that one only. They could have different rules.

In normal play, all P-positions act like zero in sums.

Sprague Grundy Theorem for normal play

$G \equiv$ impartial game played by whatever rules you like.

$*n \equiv$ single heap of size n, played as Nim.

Theorem: Every normal play impartial game G is equivalent to a unique $*n$.

Ie, there's a unique n such that

$$G + *n \text{ is a P-Position}$$

Ie,

$$G + *n = 0.$$

$n =$ "nim value of G", or "Grundy number."

Computing Grundy Numbers

$$G^{+}(G) \equiv \text{ normal play grundy number of the game } G$$

$$G^{+}(\text{empty position}) = 0$$

Suppose

$$G = \{H_1, H_2, \ldots\}, \text{ a game with options } H_1, H_2, \ldots$$

Then

$$G^{+}(\mathrm{G}) = \mathrm{mex}(\{G^{+}(H_1), G^{+}(H_2), \ldots\})$$

Here mex = "minimal excludant."

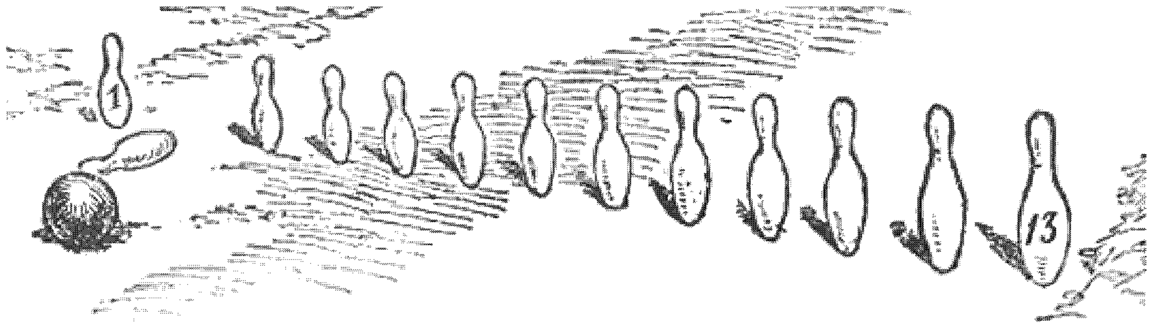
For example:

$$\mathrm{mex}(\{2, 0, 1, 5, 6\}) = 3$$

Grundy number of a sum = nim sum of the Grundy numbers.

Misère play:

> Define a symbol G^{-} instead, and put $G^{-}(\text{empty}) = 1$.
> It's still true that $G^{-}(G) = 0$ indicates a P-Position...
> ...but no simple sum formula in misère play :-(



Normal play Kayles

The Sprague-Grundy theorem lets us treat each isolated contiguous row of n pins is a *disguised nim heap* [Guy, Smith (1950s)]:

n	0	1	2	3	4	5	6	7	8	9	10	11
	0	1	2	3	1	4	3	2	1	4	2	7
12	4	1	2	7	5	4	3	2	1	4	6	7
24	4	1	2	8	1	4	7	2	1	8	6	7
36	4	1	2	3	1	4	7	2	1	8	2	7
48	4	1	2	8	1	4	7	2	1	4	2	7
60	4	1	2	8	1	4	7	2	1	8	6	7
72	4	1	2	8	1	4	7	2	1	8	2	7
84	4	1	2	8	1	4	7	2	1	8	2	7

To retain the championship of Sleepy Hollow, Rip should knock down pin number 6. This divides the pins into groups of size 1, 3 and 7 [...] To have won the game from the start, the little Man of the Mountain should have knocked down pin No. 7, so as to have divided the pins into two groups of six each.

Some results in Nim Sequence Calculation

Game code notation for the *rules* of a game played with heaps of tokens: $.\mathbf{d_0 d_1 d_2}\ldots$

If $\mathbf{d_i} = 2^a + 2^b + 2^c + \cdots$, you're allowed to remove i tokens from a heap provided you leave the remaining tokens of that heap in a or b or c or ... nonempty heaps.

For example, Kayles = $.\mathbf{77}$, Nim = $.\mathbf{333}\ldots$.

Game	Period Length	Last Exceptional Value	When
Kayles	12	70	1950s
.156	349	3478	1960s
.16	149459	105351	1980s
.376	4	2268247	1980s
.106	328226140474	465384263797	2000s

Is the nim sequence of every such game ultimately periodic?

The Big Picture

Impartial Games

normal play *misere play*

Sprague-Grundy Theory (1930s)

(*nim sequences*)

Conway's Genus Theory (1976)

Tame Games (*nim-like play*)

Wild Games (*the great unknown*)

Some Writers on Wild Games

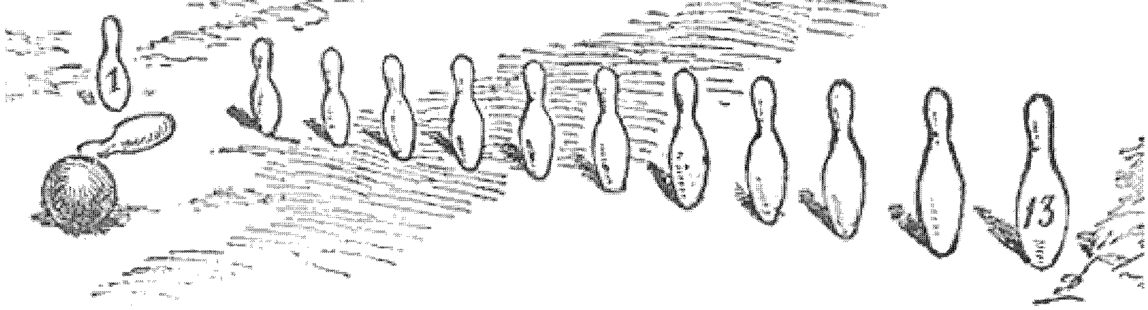
"This recondite analysis may be recommended to any mathematician [...] It is reasonably clear to me that the general case involves an infinity of expanding moduli. The results cannot be taken very far." [TR Dawson, *Caissa's Wild Roses*, 1935, writing about the game **.137**]

"The necessary tables [for outcome calculations] would be of astronomical size, even for small [positions]..." [Grundy and Smith, 1956]

"It would become intolerably tedious to push this sort of analysis much farther, and I think there is no practicable way of finding the outcome [of Grundy's game] for much larger n..." [Conway, *On Numbers and Games*, 1976]

"You musn't expect any magic formula for dealing with such positions..." [Winning Ways, 1st edition, 1983]

"Much harder to analyze." [Mathworld]



Misere play Kayles

Solution discovered by William L. Sibert (published 1992).

Follow the normal play algorithm, except *DONT* move to

$$E(5)\ E(4,1),$$

$$E(17,12,9)\ E(20,4,1),\ \text{or}$$

$$25\ E(17,12,9)\ D(20,4,1).$$

You *MAY* move to

$$D(5)\ D(4,1),$$

$$E(5)\ D(4,1),$$

$$D(9)\ E(4,1),$$

$$12\ E(4,1),$$

$$E(17,12,9)\ D(20,4,1),\ \text{or}$$

$$25\ D(9)\ D(4,1)$$

instead.

"Sibert's remarkable *tour de force* [analysis of misère Kayles] raises once again the question: are misère analyses really so difficult?" [Winning Ways, 2nd edition, 2003]

Why is misère play difficult?

(1) No global simplication rules such as $G+G=0$.

(2) Outcome class indeterminancy for sums

Form of summands $G+H$	Normal result	Misère result N example	Misère result P example
$N+N$	P iff same Grundy number	$0+0=0$	$2+2=:4$
$N+P$	N	$2+1=3$	$0+1=1$
$P+P$	P	$1+1=0$	$:4+\ :4=:8$

(3) Game trees don't simplify much:

Normal play: Only $n+1$ games "born" by day n $\{*0,*1,\ldots,*n\}$

Misere play: Roughly $2^{4171780}$ games born on day 6

(4) Number of completely analyzed wild games in the literature

[1980] fewer than 5

[1995] fewer than 10

Some small misere game trees

[from ONAG]

THE MISÈRE GRUNDY VALUE 141

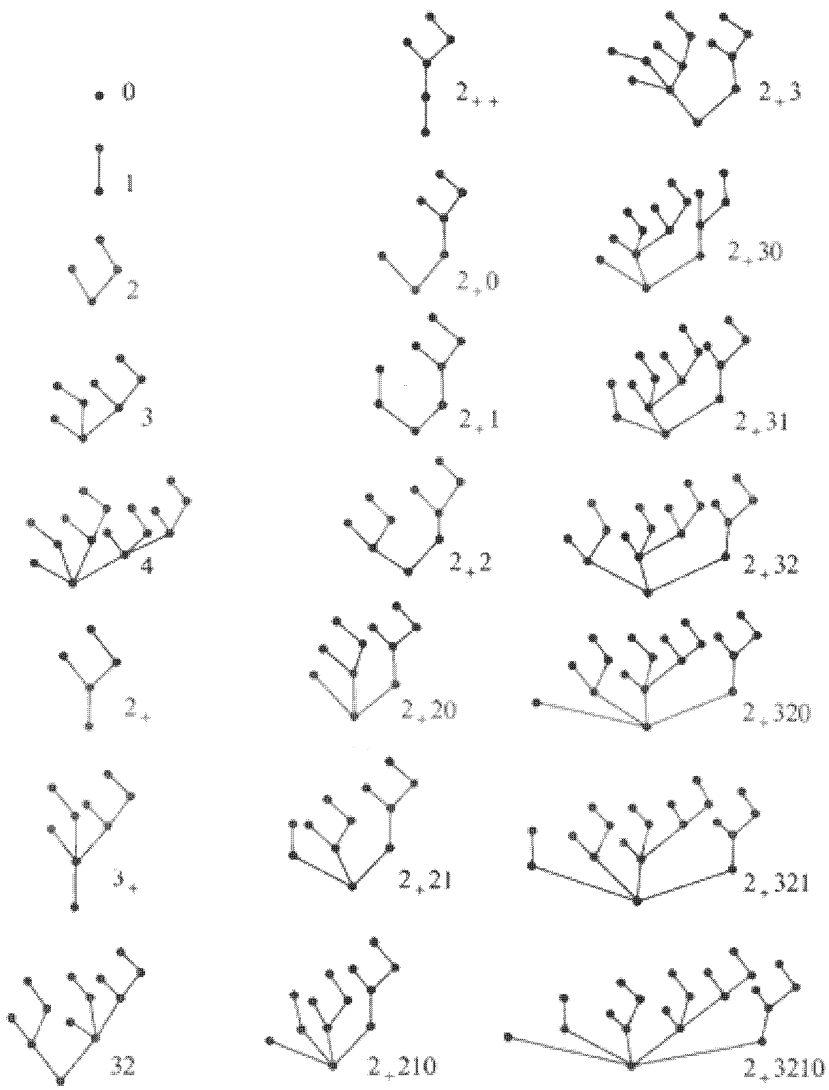


FIG. 32. The reduced games of length at most 4.

Conway's Genus Symbol

Game G

$$\text{genus}(G) = g^{g_0 g_1 g_2 \cdots}$$

where

$$\begin{aligned}
g &= \mathsf{G}^{+}(G) = \text{ normal play Grundy number} \\
g_0 &= \mathsf{G}^{-}(G) = \text{ misere play Grundy number of } G \\
g_1 &= \mathsf{G}^{-}(G + *2) = \text{ misere play Grundy number of } G + *2 \\
g_2 &= \mathsf{G}^{-}(G + *2 + *2) \text{ etc}
\end{aligned}$$

In general g_n is the G^{-}-value of the sum of G with n other games all equal to [the nim-heap of size] 2.

The "exponent" part of the genus symbol is eventually periodic of length 2.

Tame genera = genera of misere nim positions:

nim position	$G + G$	$*0$	$*1$	$*2$	$*3$	$*4$	$\cdots$
tame genus	0^{02}	0^{120}	1^{031}	2^{20}	3^{31}	4^{46}	$\cdots$

Wild genera (everything else, for example):

$$2^{1420}, 5^{061}, 4^{461}, 2^{202020431}, \ldots$$

Just one *one* wild genus amongst the positions of a game usually renders a complete analysis hopeless.

Adders

[Dean Allemang]

The *adder* games $:a$ for $a \geq 0$ are defined by setting $:0 = *0$, $:1 = *1$, and $:2 = *2$ (the misère nim heaps of size 0, 1, and 2, respectively). In general

$$:(2a) = \underbrace{:2 + :2 + \cdots + :2}_{a}$$

and

$$:(2a+1) = :(2a) + :1$$

The genus(G) symbol faithfully encodes the outcomes of all sums of G with adders.

Example:

$$\text{If genus}(G) = 2^{1420}, \text{ say}$$

$$\text{then } G + :a \text{ is a P position for } a = 1, 6, 10, 14, 18, \ldots$$

Adders play a fundamental role in misere play, similar in some ways to nim heaps in normal play.

Adding adders

In terms of addition,

$$
\begin{aligned}
:a + :b &= :(a+b) \text{ except that} \\
:a + :b &= :(a+b-2) \text{ if } a \text{ and } b \text{ both odd.}
\end{aligned}
$$

In terms of options,

$$
\begin{aligned}
:0 &= \{\} \\
:1 &= \{:0\} \\
:2 &= \{:0, :1\} \\
:3 &= \{:0, :1, :2\} \\
:4 &= \{:2, :3\} \\
:5 &= \{:2, :3, :4\} \\
:6 &= \{:4, :5\} \\
:7 &= \{:4, :5, :6\} \\
:8 &= \{:6, :7\} \\
:9 &= \{:6, :7, :8\} \\
:10 &= \{:8, :9\} \\
:11 &= \{:8, :9, :10\} \\
\cdots & \quad\ \cdots
\end{aligned}
$$

In terms of genera:

$$
\begin{aligned}
\text{genus}(:0) &= 0^{1202\cdots} = 0^{120} \\
\text{genus}(:1) &= 1^{0313\cdots} = 1^{031} \\
\text{genus}(:2) &= 2^{2020\cdots} = 2^{20} \\
\text{genus}(:3) &= 3^{3131\cdots} = 3^{31} \\
\text{genus}(:4) &= 0^{0202\cdots} = 0^{02} \\
\text{genus}(:5) &= 1^{1313\cdots} = 1^{13} \\
\text{genus}(:6) &= 2^{2020\cdots} = 2^{20} \\
\text{genus}(:7) &= 3^{3131\cdots} = 3^{31} \\
\cdots & \quad\ \cdots
\end{aligned}
$$

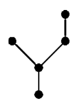
The last four entries repeat with period 4.

Pretending, I: Local indistinguishability

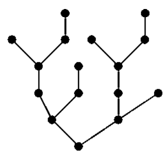
Games G, H, T

We say T *distinguishes* between G and H if $G+T$ and $H+T$ have different outcomes.

Example: Suppose we're studying **.123**. Its normal play nim sequence period has length 5. In misere play, we get trees, for example

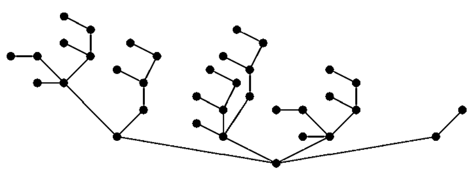


The heap h_6 in **.123** is $2_+ = \{2\}$, a game of genus 0^{02}.



The heap h_{11} in **.123**, also of genus 0^{02}.

One game T—obtained via computer search—that distinguishes h_6 and h_{11} is



The non-**.123** position T distinguishes between h_6 and h_{11}.

We have

$$\text{genus}(h_6 + T) = 0^{20}, \text{ while}$$

$$\text{genus}(h_{11} + T) = 0^{0520}.$$

The sum h_6+T is a misère N-position, while $h_{11}+T$ is a P-position.

Surprising fact—no such T actually ever occurs as a sum in .**123**!

Idea: *pretend* h_{11} can be treated as an h_6.

Pretending, II

Take the heap of size 9 in $.\mathbf{123}$.

$$\text{genus}(h_9) = 1^{20}$$

$$\text{genus}(h_9 + h_9) = 0^{02}$$

$$\text{genus}(h_9 + h_9 + h_9) = 1^{13}$$

$$\text{genus}(h_9 + h_9 + h_9 + h_9) = 0^{02}$$

$$\text{genus}(h_9 + h_9 + h_9 + h_9 + h_9) = 1^{13}$$

So whereas in **normal play** we've got

$$h_9 + h_9 = 0,$$

In misère play, we the have weaker, **.123**-specific simplification rule:

$$h_9 + h_9 + h_9 + h_9 \to h_9 + h_9$$

In **misere play**, a *pretending statement* $X_{p,s}$ asserts that for a *specific* position X,

$$\text{o}(\underbrace{X + X + \ldots + X}_{p+s \text{ copies}} + U) = \text{o}(\underbrace{X + \ldots + X}_{s \text{ copies}} + U)$$

for all games U that **actually arise as sums** according to the rules of the game under analysis. ($p \geq 1$, $s \geq 0$)

[Allemang: "Generalized genus sequence"]

Experimental Results

(1) Found hundreds of new complete analyses for wild games

(2) *Quaternary games* (code digits **0, 1, 2**, and **3** only) seem to be particularly susceptible to pretending.

Moral: Don't try this without a computer

Pretending solution for $.\mathbf{123}$

Notation: $X_{p,s}$ means that we can pretend

$$\underbrace{X + X + \ldots + X}_{p+s \text{ copies}} = \underbrace{X + \ldots + X}_{s \text{ copies}}.$$

	1	2	3	4	5
0+	: 1	: 0	: 2	: 2	: 1
5+	$C_{1,1}$	: 0	$A_{2,1}$	$B_{2,2}$	: 1
10+	$C_{1,1}$	: 0	$A_{2,1}$	$B_{2,2}$	: 1
15+	...				

The **0.123** adder approximator

C	A	B
: 4	: 2	:5

gives the correct genus for all twenty four small sums of A, B, and C with the following five exceptions:

$$\begin{aligned}
genus(B) &= 1^{20}\\
genus(A) &= 2^{1420}\\
genus(A+B) &= 3^{02}\\
genus(A+A) &= 0^{120}\\
genus(A+A+B) &= 1^{20}
\end{aligned}$$

Pretending solution for $\mathbf{.351}$

	1	2	3	4	5	6	7	8
0+	$:1$	$:2$	$:1$	$:2$	$:4$	$A_{2,2}$	$:4$	$:2$
8+	$B_{2,1}$	$:2$	$B_{2,1}$	$:2$	$:4$	$A_{2,2}$	$:4$	$:2$
16+	...							

Adder approximator:

A	B
$:5$	$:5$

Exceptions for the approximator:

Position	True Genus	Adder Approximator
A	1^{20}	1^{13}
B	1^{02}	1^{13}
$A+B$	0^{20}	0^{02}
$A+2B$	1^{20}	1^{13}

Pretending in the Quaternary Period

(1) *Subtraction game*: Particularly simple games given by a subtraction set, for example $\{1, 4, 7\}$, (game code $=$ **.3003003**).

(2) The normal play nim sequence of a subtraction game always periodic. For example, for $\{1, 4, 7\}$:

	0	1	2	3	4	5	6	7
0	0	1	0	1	2	0	1	2
8+	0	1	0	1	2	0	1	2
16+	...							

(3) The misere play of a subtraction game is always tame.

(4) The code of such a game involves digits $\mathbf{0}$ and $\mathbf{3}$ only.

(5) Quaternary games involve code digits $\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}$ only

(6) *Quaternary games are often wild, but succumb to pretending* (Experimental result)

(7) $200 REWARD: **.3102**

For more information

(1) Dean Allemang's papers

(2) My stuff on the web

`http://www.plambeck.org/oldhtml/mathematics/games/misere`

(3) `thane@best.com`