Experiments in Generalizing Geometry Theorems

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1. INTRODUCTION: THE PDN THEOREM

Well-known advances in geometry have been made with experimental, or computer-aided techniques. The first was the proof of the Four-Color Conjecture, and the second was the presumed proof of the Kepler Conjecture. Less well known was Kimberling's computerized search for new special points, lines, and circles in triangles.

I will present another example of advancing geometry experimentally. First, I need to establish a starting point. About 40 years ago I found and proved a theorem about constructions on plane $\boldsymbol{n}$-gons, but it had been found and proved as early as 1908. Writers have described this Petr-Douglas-Neumann (PDN) theorem as "beautiful" and "remarkable;" agreeing with that assessment, I took it up again a few years ago.

On each side of a general plane $\boldsymbol{n}$-gon $\mathbf{A}_{t\text{-}1}$ we place isosceles triangles with apex angles $\mathbf{2\pi}\boldsymbol{t}/\boldsymbol{n}$, making another $\boldsymbol{n}$-gon, $\mathbf{A}_t$, from those apices. The theorem states that if this is done $\boldsymbol{n}-1$ times with $\boldsymbol{t} = 1,2,\ldots,\boldsymbol{n}-1$, a single point or degenerate $\boldsymbol{n}$-gon will result. It follows that $\boldsymbol{n}$-gon $\boldsymbol{n}-2$ is regular. I will illustrate in Fig. 1 with $\boldsymbol{n}=5$. In the following material, vertices of $\boldsymbol{n}$-gon $\mathbf{A}_t$ are denoted $\mathbf{A}_{t,0},\mathbf{A}_{t,1},\ldots,\mathbf{A}_{t,n\text{-}1}$. Vertex subscripts are taken modulo $\boldsymbol{n}$.

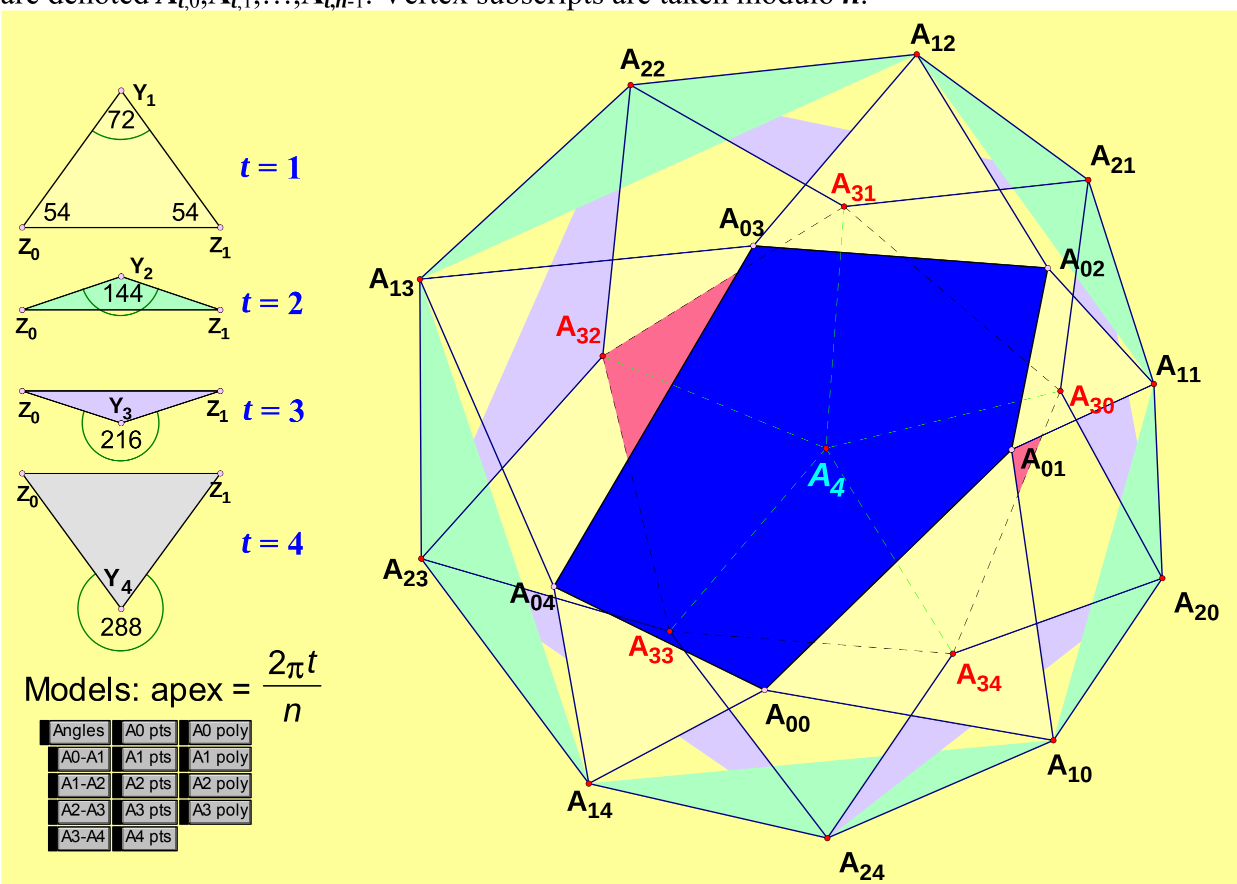


Fig. 1: Model triangles and target polygon

The triangles on the left are prototypes, or "models," for those used in the stages $\boldsymbol{t}$ of construction. The yellow model triangle corresponds to $\boldsymbol{t} = 1$, and so on down to model $\boldsymbol{t} = \boldsymbol{n}-1$. Fig. 1 shows the model triangles mounted on all four stages of the target, which is the right hand figure. According to the theorem, the angles of polygon $\mathbf{A}_3$ and the single-point property o f $\mathbf{A}_4$ are invariant with changes in $\mathbf{A}_0$. Penultimate polygon $\mathbf{A}_3$ (bright red) stays regular but may translate, rotate, and dilate. Here $\boldsymbol{n}$=5, but $\boldsymbol{n}$ can have any value greater than 2, and there will be $\boldsymbol{n}-1$ stages of construction.

I'll say a word about proofs, which have all been algebraic. We can express, for example, point $\mathbf{A}_{10}$ as a linear function of points $\mathbf{A}_{00}$ and $\mathbf{A}_{01}$, and similarly for the other vertices of $\mathbf{A}_1$. In the usual way, we represent polygons by column vectors, and each construction stage corresponds to a square matrix with exactly two nonzero elements per row. **(Eq. 1a)** This shows two of the $\boldsymbol{n}-1$ matrix multiplies.

$$\begin{bmatrix} A_{10} \\ A_{11} \\ A_{12} \\ A_{13} \\ A_{14} \end{bmatrix} = \begin{bmatrix} f_1 & 1-f_1 & 0 & 0 & 0 \\ 0 & f_1 & 1-f_1 & 0 & 0 \\ 0 & 0 & f_1 & 1-f_1 & 0 \\ 0 & 0 & 0 & f_1 & 1-f_1 \\ 1-f_1 & 0 & 0 & 0 & f_1 \end{bmatrix} * \begin{bmatrix} A_{00} \\ A_{01} \\ A_{02} \\ A_{03} \\ A_{04} \end{bmatrix}$$

$$\begin{bmatrix} A_{20} \\ A_{21} \\ A_{22} \\ A_{23} \\ A_{24} \end{bmatrix} = \begin{bmatrix} f_2 & 1-f_2 & 0 & 0 & 0 \\ 0 & f_2 & 1-f_2 & 0 & 0 \\ 0 & 0 & f_2 & 1-f_2 & 0 \\ 0 & 0 & 0 & f_2 & 1-f_2 \\ 1-f_2 & 0 & 0 & 0 & f_2 \end{bmatrix} * \begin{bmatrix} A_{10} \\ A_{11} \\ A_{12} \\ A_{13} \\ A_{14} \end{bmatrix}$$

Equation 1: Sample matrices

All $\boldsymbol{n}$-1 operations are represented as shown here:

$\mathbf{A_1}$ =	$\mathbf{M_1A_0}$ =	$\mathbf{M_1A_0}$ =	$\mathbf{Q_1A_0}$
$\mathbf{A_2}$ =	$\mathbf{M_2A_1}$ =	$\mathbf{M_2M_1A_0}$ =	$\mathbf{Q_2A_0}$
$\mathbf{A_3}$ =	$\mathbf{M_3A_2}$ =	$\mathbf{M_3M_2M_1A_0}$ =	$\mathbf{Q_3A_0}$
$\mathbf{A_4}$ =	$\mathbf{M_4A_3}$ =	$\mathbf{M_4M_3M_2M_1A_0}$ =	$\mathbf{Q_4A_0}$

Equation 1a: Matrix operations

The product $\mathbf{Q_4}$ of the four specific matrices $\mathbf{M}_{\boldsymbol{t}}$ has identical rows; this implies that every vertex of the final polygon $\mathbf{A}_4$ is the same function of the vertices of $\mathbf{A}_0$, so $\mathbf{A}_4$ is a single point. Working backwards, the previous polygon $\mathbf{A}_3$ must be regular. The constants $\boldsymbol{f_t}$ are functions only of the model angles.

For the final polygon to have a special shape, the construction must be irreversible, which in algebraic terms means that all the $\mathbf{Q}_{\boldsymbol{t}}$'s and $\mathbf{M}_{\boldsymbol{t}}$'s are singular. Also, for origin independence of the algebraic representation of the construction, any row of any $\mathbf{M}_{\boldsymbol{t}}$ must sum to 1.

2. COMPLETING THE MODELS

It's productive to draw the five triangles used in stage $\boldsymbol{t}$ = 1 assembled together, forming a pentagon (in yellow). Triangles for the other stages are also assembled into pentagons, some reflexive. The vertex common to the stage $\boldsymbol{t}$ model is denoted $\mathbf{Y}_{\boldsymbol{t}}$**.** I've varied the shades of yellow so the target and model match. The construction for stage $\boldsymbol{t}$ can be represented as follows, where $\boldsymbol{k}$ increments from 0 to $\boldsymbol{n}-1$ as we go around the model and target together.

$$\mathbf{Z}_k\,\mathbf{Z}_{k+t}\mathbf{Y}_t \rightarrow \mathbf{A}_{t-1,k+1}\mathbf{A}_{t-1,k}\mathbf{A}_{t,k}$$

Eq. 1b: Typical vertex correspondence

For example in stage $\boldsymbol{t}=2$, triangles are transferred as follows: $\mathbf{Z}_0\mathbf{Z}_2\mathbf{Y}_2 \rightarrow \mathbf{A}_{11}\mathbf{A}_{10}\mathbf{A}_{20}$, $\mathbf{Z}_1\mathbf{Z}_3\mathbf{Y}_2 \rightarrow \mathbf{A}_{12}\mathbf{A}_{11}\mathbf{A}_{21}$, etc. Similarly, the triangles used in stages $\boldsymbol{t}=3,4$ are respectively arranged around $\mathbf{Y}_3$ and $\mathbf{Y}_4$. (I'll show a complete construction key below.) Figure 2 illustrates the $\boldsymbol{n}$-1 stages and their polygons. The triangles come from generally nonadjacent model vertices but always go to adjacent target vertices.

For a constructed polygon $\mathbf{A}_{\boldsymbol{t}}$ to have a special shape, such as being regular or degenerate, stage $\boldsymbol{t}$ of construction must be irreversible, otherwise we could always find a polygon $\mathbf{A}_{t-1}$ on which the construction would give $\mathbf{A}_{\boldsymbol{t}}$, so a constructed polygon could be regular, degenerate, etc. It can be shown that the geometric criterion for irreversibility of stage $\boldsymbol{t}$ is that the construction triangles can be assembled to form a closed $\boldsymbol{n}$-gon with a continuous perimeter, like all $\boldsymbol{n}-1$ models shown here.

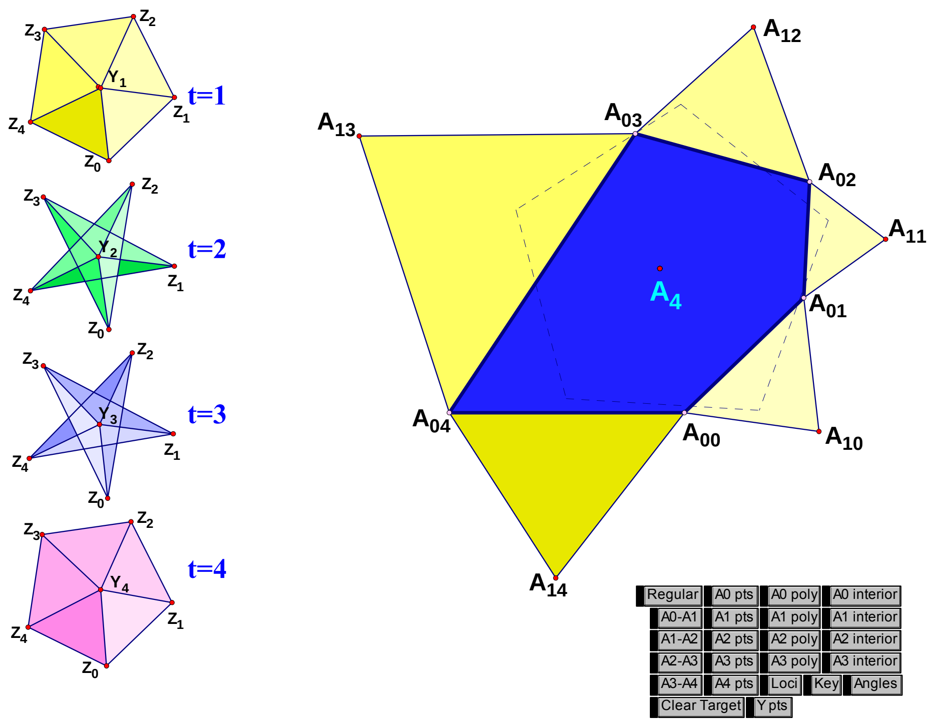


Fig. 2 All $\boldsymbol{n}-1$ "completed" models and construction stage $\boldsymbol{t}=1$ on the target.

3. THEOREMS 1 AND 2

Inspired by this type of model display, we can try moving all the $\mathbf{Y_t}$ points together away from the model centroids. Even though the construction triangles are no longer similar, $\mathbf{A_3}$ stays regular, and $\mathbf{A_4}$ is still one point. If always true, this is clearly a generalization of the PDN theorem. For a second experiment, we modify the shape of the model pentagons themselves, all together. Now $\mathbf{A}_3$ is no longer regular, but is still similar to the models. This is a second generalization, which with the arbitrary location of the $\mathbf{Y}_{\boldsymbol{t}}$, is called **Theorem 1**, appearing in the March 2003 American Mathematical Monthly [**1**]. Finally, if we move the common vertices separately, the shapes of $\mathbf{A}_{n-2}$ and $\mathbf{A}_{n-1}$ are invariant. That's a third generalization, called

Theorem 2, which I found and proved after the paper was accepted by the Monthly. (I couldn't revise the paper at that point because I had already annoyed the editor enough.)

These extensions of the original PDN theorem became possible to find only because of my introducing the ***model vs. target*** idea. This separates these theorems into two figures, making it much clearer, and allowing it to be treated in two parts. I have not seen this idea before.

4. A FURTHER GENERALIZATION; THE DISCRETE PARAMETERS

To generalize this class of theorems further, I need to introduce some constgruction parameters, shown in the green rows of Table 1. We've already discussed internal variables are $\boldsymbol{t}$ and $\boldsymbol{k}$ as well as $\boldsymbol{n}$. In the model, $\boldsymbol{p_t}$ determines which vertex is used first (that is, when $\boldsymbol{k}$=0). Next, $\boldsymbol{q_t}$ is the model vertex separation parameter for stage $\boldsymbol{t}$: if $\boldsymbol{q_t} = 1$, construction triangles come from adjacent vertices of model $\boldsymbol{t}$. If $\boldsymbol{q_t} = 2$ (the green model in Fig. 2), triangles come from vertices $\boldsymbol{k}$ and $\boldsymbol{k}+2$. Note that for Theorems 1 and 2, $\boldsymbol{q_t} = \boldsymbol{t}$, but this is not so in other results.

Qty.	Range	Meaning
$\boldsymbol{n}$	$3 \le \boldsymbol{n}$	# sides of polygons in model and target
$\boldsymbol{t}$	$1 \le \boldsymbol{t} \le \boldsymbol{n}-1$	Stage (of construction) counter
$\boldsymbol{k}$	$0 \le \boldsymbol{k} \le \boldsymbol{n}-1$	Vertex counter, increments during "scan" of model and target
$\boldsymbol{p_t}$	$0 \le \boldsymbol{p_t} \le \boldsymbol{n}-1$	On the model, which vertex is used first (that is, when k=0).
$\boldsymbol{q_t}$	$1 \le \boldsymbol{q_t} \le \boldsymbol{n}-1$	On the model, the spacing of each triangle's base vertices.
$\boldsymbol{r_t}$	$0 \le \boldsymbol{r_t} \le \boldsymbol{n}-1$	On the target, which vertex is used first (that is, when k=0).
$\boldsymbol{s_t}$	$1 \le \boldsymbol{s_t} \le \boldsymbol{n}-1$	On the target, the spacing of each triangle's base vertices.

Table 1: Discrete parameters and variables

In Theorems 1 and 2 we always use $\boldsymbol{r_t} = 0$; that is, we start (when $\boldsymbol{k}$=0) placing triangles on the target at vertex $\mathbf{A}_{t0}$. Parameters $\boldsymbol{r_t}$ and $\boldsymbol{s_t}$ are to the target as $\boldsymbol{p_t}$ and $\boldsymbol{q_t}$ are in the model. In Theorems 1 and 2 we also have $\boldsymbol{s_t} = 1$, so triangles go on adjacent target vertices in all stages $\boldsymbol{t}$. During the traverse for a given stage $\boldsymbol{t}$, as $\boldsymbol{k}$ increments, the model and target triangles may be placed on vertices as shown in the examples of Table 2. Example 1 here corresponds to Fig. 2 above. As $\boldsymbol{k}$ increments, all subscripts always increment by 1.

Example 1 (see Fig. 2)		**Example 2**
$\boldsymbol{p_t}$=0, $\boldsymbol{q_t}$=1, $\boldsymbol{r_t}$=0, $\boldsymbol{s_t}$=1.	$\boldsymbol{k}$	$\boldsymbol{p_t}$=1, $\boldsymbol{q_t}$=2, $\boldsymbol{r_t}$=3, $\boldsymbol{s_t}$=4.
$Z_0Z_1Y_t \rightarrow A_{t-1,1}A_{t-1,0}A_{t,0}$	**0**	$Z_1Z_3Y_t \rightarrow A_{t-1,2}A_{t-1,3}A_{t,3}$
$Z_1Z_2Y_t \rightarrow A_{t-1,2}A_{t-1,1}A_{t,1}$	**1**	$Z_2Z_4Y_t \rightarrow A_{t-1,3}A_{t-1,4}A_{t,4}$
$Z_2Z_3Y_t \rightarrow A_{t-1,3}A_{t-1,2}A_{t,2}$	**2**	$Z_3Z_0Y_t \rightarrow A_{t-1,4}A_{t-1,0}A_{t,0}$
$Z_3Z_4Y_t \rightarrow A_{t-1,4}A_{t-1,3}A_{t,3}$	**3**	$Z_4Z_1Y_t \rightarrow A_{t-1,0}A_{t-1,1}A_{t,1}$
$Z_4Z_0Y_t \rightarrow A_{t-1,0}A_{t-1,4}A_{t,4}$	**4**	$Z_0Z_2Y_t \rightarrow A_{t-1,1}A_{t-1,2}A_{t,2}$

Table 2: Z-A vertex correspondences

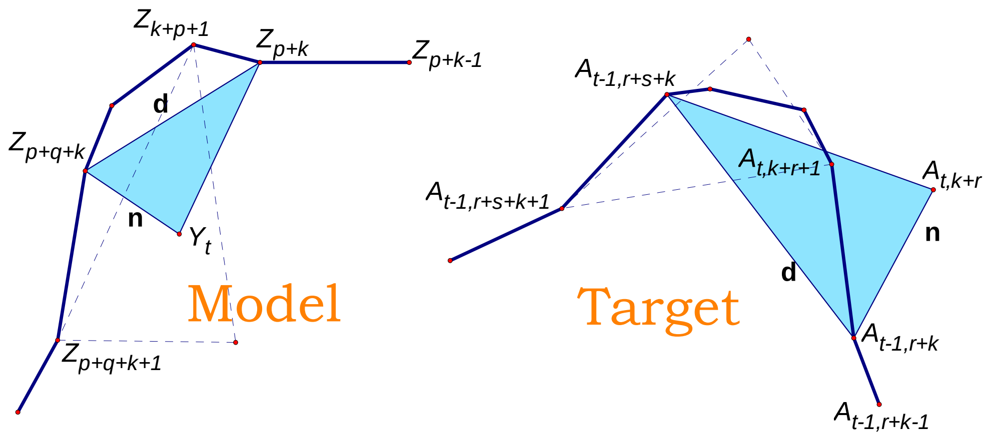
5. THE ALGEBRA

The matrices are based on the similarity of target and model triangles. In Fig. 2a, equating corresponding side ratios gives the equation shown. This figure shows a general model and its associated target triangle (in blue), for stage $\boldsymbol{t}$, at a particular place $\boldsymbol{k}$ in the traversal; it also shows the next triangle, corresponding to $\boldsymbol{k}+1$, with dashed lines. On the left, the "trailing" vertex is $\boldsymbol{Z_{p+k}}$ because the traversal starts at model vertex $\boldsymbol{p_t}$ and so far we have traversed $\boldsymbol{k}$ vertices. The leading vertex is just $\boldsymbol{q_t}$ places beyond, because that's the spacing being used at stage $\boldsymbol{t}$. Similarly, at the same time in the target we started at vertex $\boldsymbol{r}$ and have gone through $\boldsymbol{k}$ vertices there. The leading vertex is, by definition, $\boldsymbol{s}$ vertices beyond. The general transfer of a triangle from model to target for given $\boldsymbol{t}$, $\boldsymbol{k}$, $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{r}$, and $\boldsymbol{s}$ is shown in Eq. 1c ($\boldsymbol{t}$ sub-subscripts are omitted):

$$\boxed{\mathbf{Z}_{p+k}\,\mathbf{Z}_{p+q+k}\,\mathbf{Y}_t \rightarrow \mathbf{A}_{t-1,r+s+k}\,\mathbf{A}_{t-1,r+k}\,\mathbf{A}_{t,r+k}}$$

Eq. 1c: Generalized vertex correspondence

In **Fig. 2a**, similarity of the model and target blue triangles gives the equation shown.



$$\frac{A_{t,r_t+k} - A_{t-1,r_t+k}}{A_{t-1,r_t+s_t+k} - A_{t-1,r_t+k}} = \frac{Y_t - Z_{p_t+q_t+k}}{Z_{p_t+k} - Z_{p_t+q_t+k}}, \qquad 0 \le k \le n-1$$

Fig. 2a: Similar triangles and ratios of sides

That equation is solved for the new target vertex $\mathbf{A}_{t,r+k}$ in terms of the known model and the two target $\mathbf{A}_{t\text{-}1}$ vertices of the previously derived stage. With the substitution $\boldsymbol{i}=\boldsymbol{r_t}+\boldsymbol{k}$ (mod $\boldsymbol{n}$), the general matrix entry becomes that of Eq. 3.

$$(M_t)_{ij} = \frac{Y_t - Z_{p_t-r_t+i}}{Z_{p_t-r_t+q_t+i} - Z_{p_t-r_t+i}} \quad \text{if } j=i$$

$$(M_t)_{ij} = \frac{Y_t - Z_{p_t-r_t+q_t+i}}{Z_{p_t-r_t+i} - Z_{p_t-r_t+q_t+i}} \quad \text{if } j=i+s_t (\mathrm{mod}\, n)$$

$$(M_t)_{ij} = 0 \qquad \text{otherwise.}$$

Equation 3: Matrix elements

6. MORE GENERALITY; SEARCHING OVER THE DISCRETE PARAMETERS

I wanted to try to find the most general theorem of this type, limited only by the restriction that the construction be described by $\boldsymbol{p_t}$, $\boldsymbol{q_t}$, $\boldsymbol{r_t}$, and $\boldsymbol{s_t}$. I searched for parameter sets which yielded final polygons which degenerated to single points, or ones whose shape was constant with changes in $\mathbf{A}_0$. I wanted the theorems to be broad enough to include non-regular models (with the restriction that no two vertices of $\mathbf{Z}$ coincide, because of the denominators in Eq. 3). I also wanted to allow general and independent positions of the common vertices $\mathbf{Y}_t$, as well as completely general starting target polygons $\mathbf{A}_0$. Each of the four parameter sets ($\boldsymbol{p}$), ($\boldsymbol{q}$), ($\boldsymbol{r}$), and ($\boldsymbol{s}$) has has $\boldsymbol{n}$–1 members, where a member of ($\boldsymbol{p}$) or ($\boldsymbol{r}$) can take on $\boldsymbol{n}$ values, and a member of ($\boldsymbol{q}$) or ($\boldsymbol{s}$) can take on $\boldsymbol{n}-1$ values. For a complete, naïve search, we would have to go through $(\boldsymbol{n}^2-\boldsymbol{n})^{2(\boldsymbol{n}-1)}$ combinations, which even for the smallest interesting case of $\boldsymbol{n}$=5 is about 25 billion. Therefore I looked for ways to reduce the search space.

Eq. 3 shows that $\boldsymbol{p}$ and $\boldsymbol{r}$ appear only as $(\boldsymbol{p_t}-\boldsymbol{r_t})(\text{mod } \boldsymbol{n})$, so without loss of generality we can assume $\boldsymbol{p_t}=0$. Next, notice that adding a constant $\boldsymbol{c}$ to all $\boldsymbol{r_t}$ amounts to simply relabeling the target. Then if we want to investigate a certain set $(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_{\boldsymbol{n}-1})$, we just set $\boldsymbol{c} = -\boldsymbol{r}_1$ and consider only sets with $\boldsymbol{r}_1=0$. There are $\boldsymbol{n}-2$ remaining $\boldsymbol{r_t}$ parameters, each admitting $\boldsymbol{n}$ values $0 \le \boldsymbol{r_t} \le \boldsymbol{n}-1$, so ($\boldsymbol{r}$) as a whole has $\boldsymbol{n}^{\boldsymbol{n}-2}$ possible values. In each of the sets $\boldsymbol{q_t}$ and $\boldsymbol{s_t}$, $1 \le \boldsymbol{t} \le \boldsymbol{n}-1$, each member of which can take on $\boldsymbol{n}-1$ values, so the possible combinations are $(\boldsymbol{n}-1)^{\boldsymbol{n}-1}$ in number. Together, $\boldsymbol{q}$, $\boldsymbol{r}$, and $\boldsymbol{s}$ can take $(\boldsymbol{n}-1)^{2\boldsymbol{n}-2}\boldsymbol{n}^{\boldsymbol{n}-2}$ values. Even for $\boldsymbol{n}$=5 this amounts to $4^8 5^3$, over 8 million values, and for $\boldsymbol{n}$=7, it's over 36 trillion. Clearly, a general search is possible only for $\boldsymbol{n}$=5.

The main purpose of the search was to find all combinations that gave a final product matrix $\mathbf{Q}_{\boldsymbol{n}-1}$ having all rows identical. Computing this final matrix requires finding the product of $\boldsymbol{n}$ matrices, each $\boldsymbol{n}.\boldsymbol{n}$; if done numerically, this involves doing $\boldsymbol{n}^3$ multiplications and additions. Initially I considered establishing the equal-row property purely symbolically, but Mathematica took much too long (using a 1GHz PC) even for $\boldsymbol{n}$=5.

7. CONTINUOUS PARAMETERS; SEARCHING

Besides the discrete parameters of **Table 1**, the potential theorem I'm looking for has the following continuous position parameters, each one a complex number. See Table 3.

$\mathbf{Z}_{\boldsymbol{k}}$: Vertices of model, $\boldsymbol{n}$ in all, arbitrary except none may coincide.
$\mathbf{Y}_{\boldsymbol{t}}$: Common vertices of model, $\boldsymbol{n}-1$ in all. Values are arbitrary.
$\mathbf{A}_{0\boldsymbol{k}}$: Vertices of initial target, $\boldsymbol{n}$ in all. Values are arbitrary.

Table 3: Continuous parameters

In the computer search for good values of the discrete parameters, I used high-precision numeric tests with independent random values for these $6\boldsymbol{n}-2$ continuous reals. I checked the equal-row numerically, which ran at least a thousand times faster than a symbolic test would. When I got this set up correctly, it was able to run over all combinations of ($\boldsymbol{q}$), ($\boldsymbol{r}$), and ($\boldsymbol{s}$) for $\boldsymbol{n}$=5. The results were surprising: the combinations giving row equality, and therefore a successful proposition, had the unexpected rules shown in **Table 4**:

#	Rule or constraint
1.	$\boldsymbol{p_t} = 0$ $(1 \le \boldsymbol{t} \le \boldsymbol{n}-1)$ and $\boldsymbol{r}_1 = 0$ can be assumed.
2.	Set ($\boldsymbol{s}$), target vertex spacing, is the independent variable, with $\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{s})$ and $\boldsymbol{r} = \boldsymbol{r}(\boldsymbol{s})$.
3.	All $\boldsymbol{s_t}$ $(1 \le \boldsymbol{t} \le \boldsymbol{n}-1)$ are relatively prime to $\boldsymbol{n}$.
4.	The last member of the set ($\boldsymbol{s}$), $\boldsymbol{s}_{\boldsymbol{n}-1}$, can have any value $1 \le \boldsymbol{s}_{\boldsymbol{n}-1} \le \boldsymbol{n}-1$
5.	For $1 \le \boldsymbol{x} \le \boldsymbol{n}-1$, all values in $(\boldsymbol{s}_1 \ldots \boldsymbol{s}_{\boldsymbol{n}-2})$ must be either $\boldsymbol{x}$ or $\boldsymbol{n}-\boldsymbol{x}$ Examples: Valid sets for $\boldsymbol{n}$=5 are (1,4,4, $\boldsymbol{y}$), (4,4,1, $\boldsymbol{y}$), (2,3,2, $\boldsymbol{y}$), (3,3,3, $\boldsymbol{y}$), etc. For $\boldsymbol{n}$ = 7, $\boldsymbol{s}_1$ through $\boldsymbol{s}_{\boldsymbol{n}-2}$ can be any mixture of 1;6, 2;5, or 3;4.
6.	($\boldsymbol{q}$) is given by $\boldsymbol{q_t} = \boldsymbol{t}\, \boldsymbol{s_t} (\text{mod } \boldsymbol{n})$. Example for $\boldsymbol{n}$ = 5: If $\boldsymbol{s}$ = (1,4,1,3), $\boldsymbol{q}$ = (1,2,3,4)*(1,4,1,3) = (1,3,3,2). For $\boldsymbol{n}$ = 7, if $\boldsymbol{s}$ = (2,2,5,2,5,1), $\boldsymbol{q}$ = (1,2,3,4,5,6)*(2,2,5,2,5,1) = (2,4,1,1,4,6).

Table 4: Constraints on $\boldsymbol{q}$ and $\boldsymbol{s}$ for Theorem 3

Limiting the search with these constraints drastically cut the number of combinations to be tried, and it made the $\boldsymbol{n}$ = 7 case searchable. The final step was to find a rule for valid sets of $\boldsymbol{r_t}$

as a function of $\boldsymbol{s_t}$**.** Initially the values of $\boldsymbol{r_t}$ giving row equality, and therefore a valid hypothesis, had no apparent pattern. But knowing that its values had to be related to $\boldsymbol{s_t}$ in some logical way, I kept looking for this relation. Obviously the number of possible relationships would be huge, especially since I had no idea what one would look like.

With the above constraints, the number of $\boldsymbol{r_t}$, $\boldsymbol{s_t}$ combinations to try was now $\boldsymbol{n}^{\boldsymbol{n}-2}2^{\boldsymbol{n}-1}$, which for $\boldsymbol{n}$=7 is only 1.08 million; however each test still involved 343 multiplies and many other operations, so was still slow. I wanted to stay with prime $\boldsymbol{n}$, because according to rule 3 above, that would allow me to try more sets ($\boldsymbol{s}$).

8. COMPLETING THEOREM 3

Table 5 shows a typical table of numbers for $\boldsymbol{n}$=7 that I worked with. I selected a few rows from the much larger output of the actual computer search. In this section of the output data, $\boldsymbol{s}_1,\ldots,\boldsymbol{s}_5$ are restricted to values 1 and 6, although combinations of 2,5 and 3,4 are also valid.

#	$\boldsymbol{q}_1,\ldots,\boldsymbol{q}_6$	$\boldsymbol{r}_1,\ldots,\boldsymbol{r}_6$	$\boldsymbol{s}_1,\ldots,\boldsymbol{s}_6$	$\boldsymbol{g_t}=\boldsymbol{r_t}/\boldsymbol{s_t}$ (mod 7)
1.	1,2,3,4,5,6	0,0,0,0,0,0	1,1,1,1,1,1	0,0,0,0,0,0
2.	1,5,4,4,2,4	0,6,6,2,5,1	1,6,6,1,6,3	0,1,1,2,2,5
3.	1,5,4,4,2,6	0,6,6,2,5,3	1,6,6,1,6,1	0,1,1,2,2,3
4.	1,5,3,4,2,3	0,6,1,1,4,6	1,6,1,1,6,4	0,1,1,1,3,5
5.	1,2,4,4,2,3	0,0,5,1,4,6	1,1,6,1,6,4	0,0,2,1,3,5
6.	6,5,4,4,5,3	0,0,0,3,3,0	6,6,6,1,1,4	0,0,0,3,3,5
7.	1,2,3,3,5,5	0,0,0,4,1,0	1,1,1,6,1,2	0,0,0,3,1,5
8.	1,5,3,3,5,5	0,6,1,5,2,1	1,6,1,6,1,2	0,1,1,2,2,5
9.	6,2,4,4,2,2	0,1,6,2,5,6	6,1,6,1,6,5	0,1,1,2,2,5
10.	6,2,4,3,5,2	0,1,6,6,3,6	6,1,6,6,1,5	0,1,1,1,3,5

Table 5: Some successful combinations for *n* = 7

In row 1, all $\boldsymbol{s_t}$ = 1, so adjacent vertices in the target are used at every stage. All $\boldsymbol{r_t}$ = 0, so that target vertex 0 is always used first. In each stage, one additional model vertex is skipped. Row 1 in the table characterizes the PDN and Theorems 1 and 2.

Rows 1-10 show that $\boldsymbol{s}_6$ can have several values, and $\boldsymbol{q}$ is given by $\boldsymbol{q} = \boldsymbol{t}*\boldsymbol{s_t}$ in all cases. Rows 2 and 3 show that changing one member of $\boldsymbol{s}$, say $\boldsymbol{s}_3$, changes only $\boldsymbol{q}_3$, but the digits of $\boldsymbol{r_t}$ at and to the right of the changed $\boldsymbol{s}_3$ are affected.

Examining the rows, I saw that $\boldsymbol{r}_5+\boldsymbol{r}_6 = \boldsymbol{s}_5+\boldsymbol{s}_6$(mod 7). This turned out to be true for all examined cases. (In rows 1 through 5, $\boldsymbol{r}_6 = \boldsymbol{g}_6\boldsymbol{s}_6$, but this relation is not always true.) At this point I still had no function for the other members of $\boldsymbol{r}$. But knowing that $\boldsymbol{q}$ is related to $\boldsymbol{s}$ by a modular product $\boldsymbol{q} = \boldsymbol{t}*\boldsymbol{s_t}$(mod $\boldsymbol{n}$), I made a guess and took the modular quotient $\boldsymbol{g_t} = \boldsymbol{r_t} / \boldsymbol{s_t}$ (mod 7), namely that number satisfying $\boldsymbol{r_t} = \boldsymbol{g_t}\,\boldsymbol{s_t}$ (mod $\boldsymbol{n}$). The essential (and surprising observation) was that for a given $1 \leq \boldsymbol{t} < \boldsymbol{n}$, $\boldsymbol{g_t}$ is exactly the number of members of $\boldsymbol{s}_1,\ldots\boldsymbol{s}_{t-1}$) not equal to $\boldsymbol{s_t}$.

For example, in row 5 (yellow) of Table 5, $\boldsymbol{g}_2$ =0 because $\boldsymbol{s}_1$ is not equal to $\boldsymbol{s}_2$; $\boldsymbol{g}_3$ = 2 because the set ($\boldsymbol{s}_1$,$\boldsymbol{s}_2$) has 2 members not equal to $\boldsymbol{s}_3$; $\boldsymbol{g}_5$=3 since the set ($\boldsymbol{s}_1,\ldots,\boldsymbol{s}_4$) has 3 elements not equal to 6. Although $\boldsymbol{g}_5$ = 5, this is not used because of Rule 4 above.

This rule, confirmed by numerous other cases, allows us to add the final constraint on ($\boldsymbol{r}$) to the proposition. Anticipating being able to prove this, I will call it **Theorem 3**. The full set of constraints on the discrete sets ($\boldsymbol{p}$), ($\boldsymbol{q}$), ($\boldsymbol{r}$), and ($\boldsymbol{s}$) are shown in Table 6.

Further generalizations along these lines seem to be obviated by the requirement that all construction stages be irreversible.

#	Rule or constraint
1.	$p_t=0$ ($1 \le t \le n-1$) and $r_1=0$ can be assumed w.l.o.g.
2.	Set (s) is the independent variable, with $q=q(s)$ and $r=r(s)$.
3.	All s_t ($1 \le t \le n-1$) are relatively prime to n.
4.	s_{n-1} can have any value $1 \le s_{n-1} \le n-1$ (subject to Rule 3)
5.	For $1 \le x \le n-1$, all values in the set ($s_1 \ldots s_{n-2}$) must be either x or $n-x$ (subject to Rule 3) Example: If n = 5, valid sets are (1,4,4,y), (4,4,1,y), (2,3,2,y), (3,3,3,y), etc. For n = 7, s_1 through s_{n-2} can be any mixture of 1;6, 2;5, or 3;4.
6.	(q) is given by $q_t = t\, s_t$ (mod n). Example for n = 5: If s = (1,4,1,3), q = (1,2,3,4)*(1,4,1,3) = (1,3,3,2). For n = 7, if s = (2,2,5,2,5,1), q = (1,2,3,4,5,6)*(2,2,5,2,5,1) = (2,4,1,1,4,6).
7. new	For $1 \le t \le n-2$, $r_t = g_t\, s_t$ (mod n), where g_t is the number of elements in ($s_1,\ldots s_{t-1}$) which are not equal to s_t. Example: s = (2,2,5,2,5,4) ➝ g = (0,0,2,1,3,5) ➝ r = (0,0,3,2,1,x). (Rule 8 gives r_6.)
8. new	The last member of the set (r), r_{n-1}, is found from $r_{n-1}+r_{n-2} = (s_{n-1}+s_{n-2})$(mod n). Example: $r_{n-1}= s_{n-1}+s_{n-2}-r_{n-2}$ = (5+4–1)(mod 7) = 1 = r_6 .

Table 6: Constraints on *q, r,* and *s* for Theorem 3.

There are other expressions for Rule 7, but I have found none simpler than the above. It took me much longer to find these unexpected constraints, especially Rules 7 and 8, than this description implies. Discovery involved staring at these tables of numbers for a long time; I kept thinking of the movie version of John Nash. I'm unclear as to why this strange set of rules holds, and exactly what it means geometrically. In particular, it seems odd that the rule for r_{n-1} is different from the rule for r_1 through r_{n-2}. It's possible that rules 7 and 8 amount only to relabeling the model and/or the target in certain ways, but I have not seen this yet.

In return for this complexity, we have a good PDN generalization, namely **Theorem 3**, which holds for any $n>2$. For prime n, the restricted set (s) can have $2^{n-3}(n-1)^2$ values, with (q) and (r) fixed functions of (s). (The first and last s's can have any value $1 \le s_1, s_{n-1} \le n-1$, and the other $n-3$ s_t's can each have two values dependent on s_1.) As pointed out previously, the theorem also involves $3n-1$ continuous complex parameters. Even for n=7, the independent discrete parameters number 576 and there are 64 independent reals. **Table 6a** gives the number of combinations to be tested under various assumptions. (Having found the constraints for Theorem 3, searching is no longer necessary; the last column gives the number of parameter combinations making the theorem valid.)

Search type	Formula	n=5	n=7
Over all (p),(q),(r),(s)	$n^{2n-2}(n-1)^{2n-2}$	$2.56*10^{10}$	$3.01*10^{19}$
Restrictions on (p),(r)	$n^{n-2}(n-1)^{2n-2}$	$8.19*10^{6}$	$3.66*10^{13}$
With Theorem 3 rules	$(n-1)^2 2^{(n-3)}$	64	576

Table 6a: Number of combinations to be tried

9. EXAMPLE OF THEOREM 3 FOR n=7

Table 7 gives the parameters for this example. As before, t is the construction stage variable. Rule 4 in Table 6 says that $s_{n\text{-}1}$ can have any value. Rule 5 is obeyed because the other s are either 2 or 7–2. Rule 6 is followed because $q_1=s_1$, $q_2=2s_2$ (mod n), etc. Including the dependent variables in (g), note that the parameters obey Rule 7, $r_t=g_t s_t$(mod n), except for r_6 which is given by Rule 8.

t=	1	2	3	4	5	6
p_t	0	0	0	0	0	0
q_t	2	3	6	6	4	3
r_t	0	5	2	3	3	4
s_t	2	5	2	5	5	4
g_t	0	1	1	2	2	5

Table 7: p,q,r,s for the n=7 Figure.

Fig. 3 shows only the $\mathbf{A}_0$ and $\mathbf{A}_5$ polygons, and the final As claimed, $\mathbf{A}_5$ is similar to the model, and stays so if we change $\mathbf{A}_0$. If we move $\mathbf{Y}_1$ or $\mathbf{Z}_6$, all intermediate polygons change but $\mathbf{A}_5$ is unchanged in shape.

I don't yet have a proof of this proposition, but it's almost certainly correct, and I'm working to prove it. My proof of **Theorem 1** proceeds by induction on partial matrix products, computed and examined with the help of Mathematica. Proof of **Theorem 3** will probably also be computer-aided.

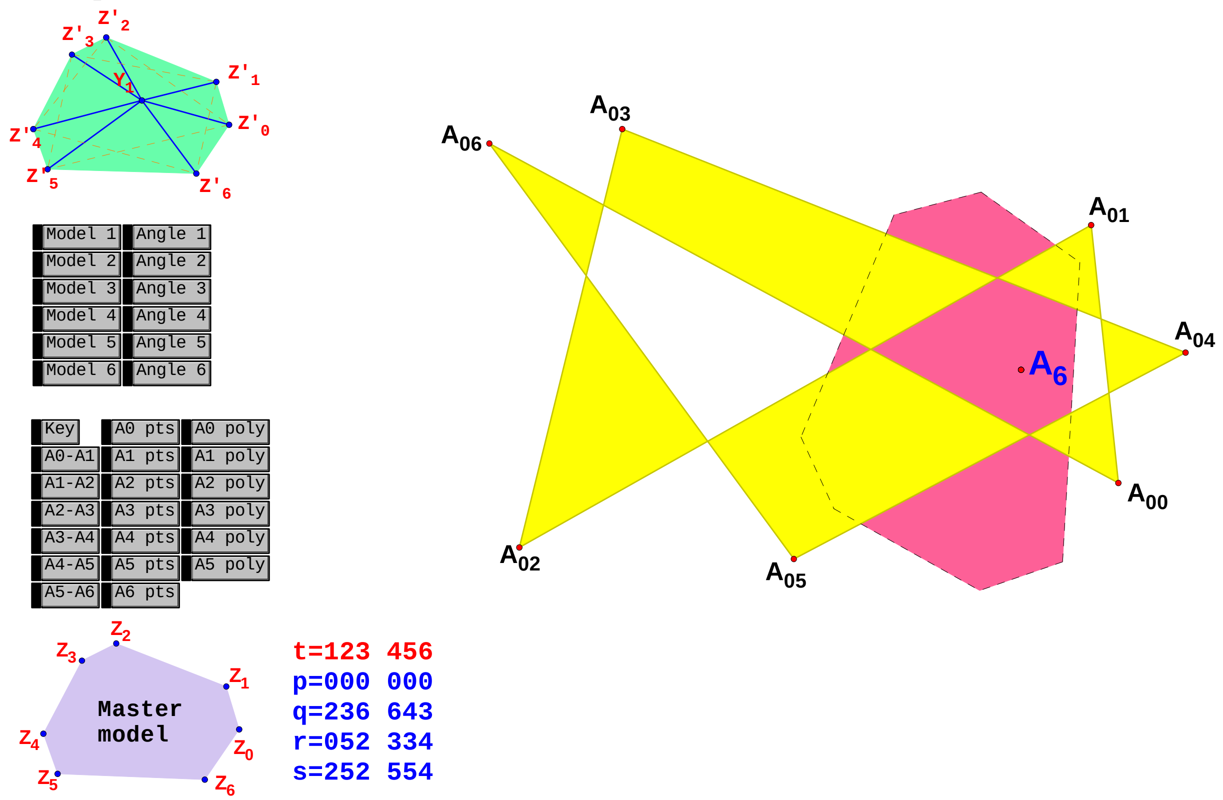


Fig. 3: Theorem 3 for p_t, q_t, r_t, s_t as shown.

10. REMARKS

Table 8 summarizes the theorems and shows their current status.

Name	Models Z_t	Common Vertices Y_t	Target vertices for triangles	Current status
PDN	Regular	$Y_t = Z_{\text{centroid}}$	$s_t = 1$	Published in 1908
Theorem 1	Irregular	$Y_t \neq Z_{\text{centroid}}$ $Y_t = Y_u$	$s_t = 1$	Proved, Monthly, Mar. 2003
Theorem 2	Irregular	$Y_t \neq Z_{\text{centroid}}$ $Y_t \neq Y_u$	$s_t = 1$	Proved but not published
Theorem 3	Irregular	$Y_t \neq Z_{\text{centroid}}$ $Y_t \neq Y_u$	$s_t \neq 1$, constraints on ($\boldsymbol{q}$), ($\boldsymbol{r}$), ($\boldsymbol{s}$)	Not proved or published

Table 8. Summary of theorems

It is possible that Theorem 3 does not really contain new geometry, but can be derived from Theorem 2 by relabeling the model and/or target, perhaps with some algebraic substitutions. Another point is that without the computer search, I would not know whether the $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{r}$, $\boldsymbol{s}$ paradigm contained further geometric truths than I have reported here. As things stand, Theorem 3 is the last generalization along this line; however I have found other similar propositions, which I intend to prove and publish eventually. This field seems to be very fertile in new results.

11. SOME RELATED QUESTIONS IN COMBINATORIAL GEOMETRY

1. What is the number $d_{m,n}$ of “topologically distinct” $\boldsymbol{n}$-gons obtainable by merging sets of two or more adjacent vertices of an $\boldsymbol{m}$-gon, with $n<m$? Concepteually, put a black bead on every edge to be collapsed and a white bead on the edges between the vertices kept separate. The $\boldsymbol{n}$-gon count can be shown to be equivalent to the number of two-color bracelets (necklaces allowing reversal) having $m-n$ black and $\boldsymbol{n}$ white beads; $d_{m,n}$ is given by A052307 in the EIS. Examples are $d_{5,3}$=2, $d_{6,4}$=3, and $d_{12,5}$=38, so there are two distinct ways to reduce a pentagon to a triangle, three different reductions of a hexagon to a quadrilateral, etc. (Thanks to Claude Chaunier for showing me this.) These reductions create new, strange, perhaps ugly theorems for $\boldsymbol{n}$-gons from constructions for $\boldsymbol{m}$-gons, with $n<m$.

2. For n=5, the number of line intersections or crossings, excluding vertices, can be 0,1,2,3, and 5. I denote this set $(c_5) = (0,1,2,3,5)$, so the cardinality of the set is $|c_5| = 5$. For the 7-gon in Fig. 3, $c_x = 8$, so $8 \in (c_7)$. What is $|c_7|$? For a given $\boldsymbol{n}$, what values can appear in the set (c_n), and is the integer sequence $|c_n|$ in the EIS? Where?

3. Count the number e_n of topologically distinct polygons of $\boldsymbol{n}$ sides. In my definition, a topological change is defined by passing a vertex through a line. Mirroring is not a change. For n = 3,4,5, e_n = 1,2,6 respectively. The self-crossing polygon of Fig. 3 is one of perhaps several dozen 7-gons. This is probably in the EIS, but where? Is there a better definition of “distinct?”

4. Is there an $\boldsymbol{n}$-gon in $\mathbf{R}^3$ whose projections into $\mathbf{R}^2$ assume all distinct $\boldsymbol{n}$-gons? For what values of $\boldsymbol{n}$ is this possible, in addition to the obvious cases n = 3 and 4? For how large an $\boldsymbol{n}$ are experiments feasible? I’ve always wanted to pose a difficult new problem. Is this one? ■

REFERENCES

1. S. B. Gray, Generalizing the Petr-Douglas-Neumann theorem on $\boldsymbol{n}$-gons, *American Mathematical Monthly* **110** (2003), 210-227.