Accuracy of Projection Methods for the Incompressible Navier-Stokes Equations

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Abstract. Numerous papers have appeared in the literature over the past thirty years discussing projection-type methods for solving the incompressible Navier-Stokes equations. A recurring difficulty encountered is the proper choice of boundary conditions for the auxiliary variables in order to obtain at least second order accuracy in the computed solution. A further issue is the formula for the pressure correction at each timestep. An overview of boundary condition choices that give second-order convergence for all solution variables is presented here based on recently published results by Brown, Cortez and Minion [2].

1 Introduction

Denoting by $\mathbf{u}$, the velocity, p, the pressure, and ν, the viscosity of the fluid, the incompressible Navier-Stokes equations

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \nu\nabla^2\mathbf{u} \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0 \tag{2}$$

are considered in a region $\Omega \in \mathbb{R}^n$, with boundary conditions

$$B(\mathbf{u}, p) = 0 \qquad \text{on} \quad \partial\Omega. \tag{3}$$

Typical boundary conditions might be those for a solid wall:

$$\mathbf{u}\cdot\hat{\mathbf{n}} = 0 \quad \text{“no flow”} \tag{4a}$$
$$\mathbf{u}\cdot\hat{\tau} = 0 \quad \text{“no slip”}. \tag{4b}$$

where local normal and tangential vectors to the wall are given by $\hat{\mathbf{n}}$ and $\hat{\tau}$. Subscripts denote partial differentiation. Specifying the pressure, its normal derivative, or a combination of the two at outflow is also a possibility:

$$\alpha p + \beta\hat{n}\cdot\nabla p = g. \tag{5}$$

For a discussion of allowable boundary conditions see, e.g. [9].

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2 Projection methods

Projection methods, or "fractional step" methods, as they are sometimes called, advance the momentum equation (1) and enforce the continuity condition (2) in separate steps [1,4,10,13]. These methods make use of the Hodge decomposition theorem, which states that any vector function $\mathbf{v}(\mathbf{x})$ can be decomposed into a divergence-free part $\mathbf{u}$ plus the gradient of a scalar potential ϕ, i.e.

$$\mathbf{v}(\mathbf{x}) = \mathbf{u}(\mathbf{x}) + \nabla\phi(\mathbf{x}) \tag{6}$$

with $\nabla \cdot \mathbf{u} = 0$, where furthermore, using a suitable inner product, $(\mathbf{u}, \nabla\phi) = 0$, i.e. the two parts are orthogonal. In order for the decomposition to be unique, boundary conditions must be specified as well. For the purposes of this paper, we choose to specify the normal component of the velocity, i.e.

$$\hat{\mathbf{n}} \cdot \mathbf{u} = u_b. \tag{7}$$

Thus, the divergence-free part of an arbitrary vector $\mathbf{v}$ can be obtained by a projection onto the orthogonal subspace of divergence-free vectors by removing the gradient of an appropriately chosen scalar potential ϕ. The notation

$$\mathbf{u} = \mathbf{P}(\mathbf{v}) \tag{8}$$

is sometimes used to express this projection.

Using this information, one is naturally led to an approach whereby an approximation to the momentum equation

$$\mathbf{u}^*{}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla q = \nu\nabla^2\mathbf{u}^*, \tag{9}$$

is advanced for some interval of time $t_o <= t <= t_1$, and the divergence-free velocity is computed when needed using the projection

$$\mathbf{u}(t) = \mathbf{P}(\mathbf{u}^*(t)). \tag{10}$$

Here $\nabla q(\mathbf{x}, t)$ is some approximation to the pressure gradient, which may even be zero (see e.g. [10]). As a practical matter, the projection is effected by deriving an elliptic equation as follows. Applying the Hodge decomposition theorem, we can write

$$\mathbf{u}^* = \mathbf{u} + \nabla\phi. \tag{11}$$

Taking the divergence of (11) and applying (2), results in the elliptic constraint equation for ϕ

$$\nabla^2\phi = \nabla\cdot\mathbf{u}^*. \tag{12}$$

The pressure can be recovered at any time using the formula

$$\nabla p = \nabla(q + \phi_t) - \nu\nabla^2\nabla\phi, \tag{13}$$

which is derived by substituting (11) into (9) and comparing with the original momentum equation (1). Common practice for projection methods is to advance (9) for a single timestep, compute $\mathbf{u}$ at the end of the timestep using the projection, and then replace $\mathbf{u}^*$ with this new value of $\mathbf{u}$ at the beginning of the next timestep. When $\mathbf{u}^*$ is reset to $\mathbf{u}$ at the end of each timestep, it always stays relatively close to $\mathbf{u}$ during the computation. Various projection methods discussed in the literature differ in their approximation of the advective terms $(\mathbf{u} \cdot \nabla)\mathbf{u}$, the approximation used for q, whether (9) is advanced explicitly or implicitly, and how the pressure gradient update formula (13) is approximated.

An alternative class of methods does not reset $\mathbf{u}^*$ at the end of each timestep, but allows it to evolve over the period of the computation. Such methods, known variously as "magnetization", "impulse" or "gauge" methods have been developed by Buttke [3], Cortez [5,6], E and Liu [7,8], Recchioni and Russo [11] and Summers and Chorin [12]. One advantage of such methods is that the approximation $q = 0$ can be used, and the pressure never need be determined unless pressure-dependent boundary conditions are required. Note also that the demonstrated success of these methods indicates that the length of time that the approximate momentum equation (9) can be integrated need not be restricted to a single timestep. These methods will not be discussed further in the present paper.

3 Boundary conditions when pressure term is included in predictor

Boundary conditions are required on ϕ in order to solve (12). In addition, boundary conditions on $\mathbf{u}^*$ are required if (9) is to be advanced implicitly and also to compute the right-hand side of (12). Since both $\mathbf{u}^*$ and ϕ are auxiliary variables, the original problem formulation does not tell how to set their boundary values. However it is clear that any boundary conditions that are specified must satisfy (11) as a constraint, i.e.

$$\nabla\phi|_{\partial\Omega} = (\mathbf{u}^* - \mathbf{u})|_{\partial\Omega}. \tag{14}$$

Since both $\mathbf{u}^*$ and $\mathbf{u}$ are known at the point that the elliptic equation (12) is solved, and given our choice for the boundary condition form (7), the appropriate boundary condition for this equation is therefore

$$\hat{\mathbf{n}} \cdot \nabla\phi|_{\partial\Omega} = \hat{\mathbf{n}} \cdot (\mathbf{u}^* - \mathbf{u})|_{\partial\Omega}. \tag{15}$$

The boundary conditions on $\mathbf{u}^*$ are more problematic since they are needed before ϕ is computed. As reported in [2], the correct choice depends on the approximation for ∇q used in (9). Bell, Colella and Glaz [1] advance (9) using Crank-Nicolson for time integration, approximate the advective terms with a Godunov procedure, and use the time-centered pressure gradient from the previous timestep to approximate ∇q:

$$\frac{(\mathbf{u}^* - \mathbf{u}^{\mathbf{n}})}{\Delta t} + ((\mathbf{u} \cdot \nabla)\mathbf{u})^{n+\frac{1}{2}} + \nabla p^{n-\frac{1}{2}} = \frac{\nu}{2}(\nabla^2\mathbf{u}^* + \nabla^2\mathbf{u}^{\mathbf{n}}) \tag{16}$$

Here, superscripts involving n denote the time level. Since $\nabla p^{n-\frac{1}{2}} = \nabla p^{n+\frac{1}{2}} + \mathcal{O}(\Delta t)$, (16) implies that $\mathbf{u}^* = \mathbf{u}^{n+1} + \mathcal{O}(\Delta t^2)$, and so it appears plausible that a second-order method can be obtained simply by using the original specified boundary conditions for $\mathbf{u}$ as boundary conditions for $\mathbf{u}^*$, i.e.

$$\mathbf{u}^*|_{\partial\Omega} = \mathbf{u}^{n+1}|_{\partial\Omega}. \tag{17}$$

Substituting this condition into (15), it is apparent that the boundary condition for ϕ becomes particularly simple in this case as well:

$$\hat{\mathbf{n}} \cdot \nabla\phi|_{\partial\Omega} = 0. \tag{18}$$

In fact in [2], normal mode analysis is used to show that if an appropriate approximation to the pressure gradient update formula (13) is used, namely

$$\nabla p^{n+\frac{1}{2}} = \nabla p^{n-\frac{1}{2}} + \frac{1}{\Delta t}(\nabla\phi - \frac{\nu\Delta t}{2}\nabla^2\nabla\phi) \tag{19}$$

both the velocity $\mathbf{u}$ and the pressure p converge to second order in Δt.

4 The pressure gradient update formula

Implementing (19) is problematical because of the term involving $\nabla^2\nabla\phi$. Near boundaries, for example, formulation of an appropriately accurate discretization of this term can be difficult. Removing this term altogether apparently involves only an $\mathcal{O}(\Delta t)$ perturbation to the formula for $\nabla p^{n+\frac{1}{2}}$. Again, by inspection of (9), It might therefore be reasonable to expect that using the approximation

$$\nabla p^{n+\frac{1}{2}} = \nabla p^{n-\frac{1}{2}} + \frac{1}{\Delta t}\nabla\phi, \tag{20}$$

as is done in [1] and related papers, would lead to a second-order result for the velocity. It remains the case, however, that this is a first order perturbation to the formula for the pressure, so second-order convergence of the pressure would not be expected. Since the pressure and velocity are coupled by the Navier-Stokes equations, it would also seem doubtful that the velocity would be computed to second-order by this method. Surprisingly, in [2], it has been both proven using normal mode analysis and demonstrated numerically that using (20) in conjunction with (16), while it leads to only first order convergence of the pressure, does result in second-order convergence of the velocity. The first order convergence of the pressure results from a spurious mode in the pressure that converges like $\mathcal{O}(\nu\Delta t)$. This mode can only be annihilated by using the update formula for the pressure given in (19).

5 Boundary conditions when pressure term is set to zero in predictor

Provided that boundary conditions can be chosen appropriately, any convenient approximation can be used for the ∇q term in (9) since the subsequent projection step will always pick out the correct potential gradient to give a divergence-free velocity. The method proposed by Kim and Moin [10] uses $q = 0$, for example. Finding appropriate boundary conditions for $\mathbf{u}^*$ and ϕ become more difficult in this case, since it is no longer true that $\mathbf{u}^*$ is an $\mathcal{O}(\Delta t^2)$ perturbation of $\mathbf{u}$. The normal mode analysis presented in [2] suggests again that as long as the compatibility condition (14) is satisfied, second-order convergence should be possible. Since the compatibility condition is only one constraint on the boundary conditions, a convenient choice such as $\hat{\mathbf{n}} \cdot \nabla\phi = 0$ seems desirable, leading to the proposed boundary conditions

$$\hat{\mathbf{n}} \cdot \mathbf{u}^*|_{\partial\Omega} = \hat{\mathbf{n}} \cdot \mathbf{u}^{n+1}|_{\partial\Omega} \tag{21a}$$

$$\hat{\tau} \cdot \mathbf{u}^*|_{\partial\Omega} = \tau \cdot (\mathbf{u}^{n+1} + \nabla\tilde{\phi}^{n+1})|_{\partial\Omega} \tag{21b}$$

$$\hat{\mathbf{n}} \cdot \nabla\phi^{n+1}|_{\partial\Omega} = 0 \tag{21c}$$

Here, $\nabla\tilde{\phi}^{n+1}$ is an approximation to $\nabla\phi^{n+1}$. According to the normal mode analysis [2], a first order approximation to $\nabla\phi^{n+1}$ is required in order to obtain second-order convergence, and so

$$\nabla\tilde{\phi}^{n+1} = \nabla\phi^n \tag{22}$$

is used. Numerical studies using these boundary conditions, however, demonstrate only first-order convergence for the pressure in many cases even when using the improved formula

$$p^{n+\frac{1}{2}} = \phi^{n+1} - \frac{\nu\Delta t}{2}\nabla^2\phi \tag{23}$$

for the pressure. The problem with using these boundary conditions lies in the fact that the function $\mathbf{u}^*$ is not smooth near boundaries in all cases. Since $\nabla^2\phi = \nabla\cdot\mathbf{u}^*$, (23) implies that the pressure will not be smooth either near boundaries.

Rather than choosing homogeneous Neumann boundary conditions for ϕ in this case, a better idea is to choose the boundary condition for $\mathbf{u}^*$ in such a way as to guarantee smoothness of that function up to the boundary. Instead of (21a), we extrapolate $\hat{\mathbf{n}} \cdot \mathbf{u}^*$ to the boundary using at least a second-order extrapolation formula. The boundary conditions become

$$E\hat{\mathbf{n}} \cdot \mathbf{u}^*|_{\partial\Omega} = 0 \tag{24a}$$

$$\hat{\tau} \cdot \mathbf{u}^*|_{\partial\Omega} = \tau \cdot (\mathbf{u}^{n+1} + \nabla\tilde{\phi}^{n+1})|_{\partial\Omega} \tag{24b}$$

$$\hat{\mathbf{n}} \cdot \nabla\tilde{\phi}|_{\partial\Omega} = \hat{\mathbf{n}}(\mathbf{u}^* - \mathbf{u})|_{\partial\Omega} \tag{24c}$$

While an inhomogenous Neumann condition is now required for the elliptic problem for ϕ, the corresponding numerical experiments demonstrate fully second-order convergence for both the velocity and pressure, justifying the additional effort.

6 Summary of results

Table 1 summarizes the convergence results reported here and in [2]. The columns labeled "boundary conditions" indicate what the inhomogeneous terms are in the boundary conditions for $\mathbf{u}^*$ and $\hat{\mathbf{n}} \cdot \nabla\phi$. The columns labeled "difference approx." indicate the choice for q and the formula used for the pressure update or evaluation. The number shown in the columns labeled "conv. rate" are the exponent in the observed convergence rate in time for the resulting combination of boundary conditions and difference approximation.

Table 1. The convergence rate for the pressure depends upon the choice of boundary conditions and pressure updates in the projection scheme.

Boundary Conditions			**Difference Approx.**		**Conv. Rate**	
$\hat{\mathbf{n}} \cdot \mathbf{u}^*\|_{\partial\Omega}$	$\hat{\tau} \cdot \mathbf{u}^*\|_{\partial\Omega}$	$\hat{\mathbf{n}} \cdot \nabla\phi^{n+1}\|_{\partial\Omega}$	q	p update	u	p
$\hat{\mathbf{n}} \cdot \mathbf{u}^{\mathbf{n+1}}$	$\hat{\tau} \cdot \mathbf{u}^{\mathbf{n+1}}$	0	$p^{n-\frac{1}{2}}$	(20)	2	**1**
$\hat{\mathbf{n}} \cdot \mathbf{u}^{\mathbf{n+1}}$	$\hat{\tau} \cdot \mathbf{u}^{\mathbf{n+1}}$	0	$p^{n-\frac{1}{2}}$	(19)	2	2
$\hat{\mathbf{n}} \cdot \mathbf{u}^{\mathbf{n+1}}$	$\hat{\tau} \cdot (\mathbf{u}^{\mathbf{n+1}} + \nabla\phi^n)$	0	0	(23)	2	**1**
Extrapolate	$\hat{\tau} \cdot (\mathbf{u}^{\mathbf{n+1}} + \nabla\phi^n)$	$\hat{\mathbf{n}} \cdot (\mathbf{u}^* - \mathbf{u}^{\mathbf{n+1}})$	0	(23)	2	2

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