Applied Numerical Mathematics 60 (2010) 176–192

Contents lists available at ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

Comparison of bounds for V-cycle multigrid

Artem Napov *,[1], Yvan Notay [2]

Service de Métrologie Nucléaire, Université Libre de Bruxelles (C.P. 165/84), 50, Av. F.D. Roosevelt, B-1050 Brussels, Belgium

A R T I C L E   I N F O

Article history:
Received 7 August 2009
Received in revised form 14 October 2009
Accepted 9 November 2009
Available online 13 November 2009

MSC:
65F10
65N55
65F50

Keywords:
Multigrid
V-cycle
Successive subspace correction
Approximation property
Convergence analysis

A B S T R A C T

We consider multigrid methods with V-cycle for symmetric positive definite linear systems. We compare bounds on the convergence factor that are characterized by a constant which is the maximum over all levels of an expression involving only two consecutive levels. More particularly, we consider the classical bound by Hackbusch, a bound by McCormick, and a bound obtained by applying the successive subspace correction convergence theory with so-called a-orthogonal decomposition. We show that the constants in these bounds are closely related, and hence that these analyses are equivalent from the qualitative point of view. From the quantitative point of view, we show that the bound due to McCormick is always the best one. We also show on an example that it can give satisfactory sharp prediction of actual multigrid convergence.

1. Introduction

We consider multigrid methods for solving symmetric positive definite (SPD) $n \times n$ linear systems:

$$Ax = b. \tag{1.1}$$

Multigrid methods are based on the recursive use of a two-grid scheme. A basic two-grid method combines the action of a *smoother*, often a simple iterative method such as Gauss–Seidel, and a *coarse grid correction*, which involves solving a smaller problem on a coarser grid. A V-cycle multigrid method is obtained when this coarse problem is solved approximately with 1 iteration of the two-grid scheme on that level, and so on, until the coarsest level, where an exact solve is performed. Other cycles may be defined, including the W-cycle based on two recursive applications of the two-grid scheme at each level; see, e.g., [16].

When the system (1.1) stems from the discretization of an elliptic PDE, V-cycle multigrid has often optimal convergence properties; that is, the convergence is independent of the number of levels or, equivalently, of the mesh discretization parameter h. There are two classical ways for proving this. One way consists in checking the so-called smoothing and approximation properties [3,4,7,8,10,11,15]. Another possibility consists in defining an appropriate subspace decomposition and then analyze the constants involved in the successive subspace correction (SSC) convergence theory [13,14,6,18,20, 19]. So far, these approaches have only been compared (e.g., in [20]) on the basis of the regularity assumptions that an

* Corresponding author.
E-mail addresses: anapov@ulb.ac.be (A. Napov), ynotay@ulb.ac.be (Y. Notay).

1 Supported by the Belgian FNRS ("Aspirant").
2 Supported by the Belgian FNRS ("Directeur de recherches").

0168-9274/$30.00 © 2009 IMACS. Published by Elsevier B.V. All rights reserved.
doi:10.1016/j.apnum.2009.11.003

elliptic boundary value problem should fulfill in order to guarantee optimal bounds for the multigrid method applied to its finite element discretization. This allows only qualitative conclusions which are further restricted to a specific context. For instance, such a comparison does not cover V-cycle multigrid for structured linear systems [1]. In fact, a detailed comparison of the convergence theories for V-cycle is difficult because they may be (and have been) formulated diversely. There is some freedom in choosing the subspace decomposition for the SSC convergence theory and there is no unique definition of the smoothing and approximation properties.

The smoothing and approximation property ideas form the basis of the early proofs [3,4,7] of h-independent V-cycle convergence. For the case when A is SPD, the classical proof is presented in [8, Theorem 7.2.2] by Hackbusch. The convergence estimate is then characterized by the approximation property constant c_A, which is a maximum over all levels of an expression involving only two consecutive levels.

An alternative approach has been developed by McCormick in [11] (see also [10,15]). Here again, the convergence estimate depends on a constant δ which is a minimum over all levels of an expression involving two consecutive levels.

The SSC convergence theory is more recent and also more general, since by tuning the choice of the space decomposition one can prove some results for elliptic PDEs without requiring regularity assumptions [5]. The comparison with other approaches is not easy because this theory is traditionally formulated in an abstract setting. In this paper, we first develop an algebraic formulation of the theory, resulting in a bound which also depends on freely chosen quantities. Next, we justify that this degree of freedom seemingly disappears if one adds the constraint that one must be able to assess the main constant in the bound considering only two levels at a time. Note that this latter constraint is not only mandatory to develop the comparison with the other two approaches. It is also very sensible in view of a quantitative analysis, where, as we illustrate on an example, Fourier analysis setting is used to numerically calculate the bounds and compare them with the actual convergence factor.

Transferred back into the original SSC setting, the choice for which this two-level assessment is possible corresponds to the so-called a-orthogonal decomposition, which is also the decomposition that has been most extensively used when analyzing multigrid methods for the class of (H^2-)regular problems. Then, the bound depends mainly on a constant K and, in this paper, we show that the three constants c_A, δ and K are in fact closely related, namely

$$K = \max(1, c_A)$$

and

$$\delta^{-1} = c_A^{(2)},$$

where $c_A^{(2)}$ is a Hackbusch approximation property constant for the number of smoothing steps being doubled. Hence the three approaches are qualitatively equivalent, in the sense that they simultaneously succeed or fail to prove optimal convergence. From the quantitative point of view, it further turns out that McCormick's bound is the best one.

The reminder of this paper is organized as follows. In Section 2, we state the general setting of this study and gather the needed assumptions. In Section 3, we develop our algebraic variant of the SSC theory and recall the results of Hackbusch and McCormick. The comparison is performed in Section 4, and an example is analyzed in Section 5.

1.1. Notation

Let I denote the identity matrix and O the zero matrix. When the dimensions are not obvious from the context, we write more specifically I_m for the $m \times m$ identity matrix, and $O_{m\times l}$ for the $m \times l$ zero matrix.

For any real matrix B, $\mathcal{R}(B)$ is the range of B and $\mathcal{N}(B)$ is its null space. For any square real matrix C, $\rho(C)$ is its spectral radius (that is, its largest eigenvalue in modulus), $\|C\| = \sqrt{\rho(C^TC)}$ is the usual 2-norm and $\|C\|_{\mathcal{F}} = \sqrt{\sum_{i,j} C_{ij}^2}$ the Frobenius norm. For a SPD matrix D, $\|v\|_D = (v^TDv)^{1/2} = \|D^{1/2}v\|$ is the associated D-norm of a vector v (if $D = A$, it is also called energy norm) and

$$\|C\|_D = \max_v \frac{\|Cv\|_D}{\|v\|_D} = \left\|D^{1/2}CD^{-1/2}\right\|$$

is the induced matrix D-norm.

2. General setting

We consider a multigrid method with $J+1$ levels ($J \geqslant 1$); index J refers to the finest level (on which the system (1.1) is to be solved), and index 0 to the coarsest level. The number of unknowns at level k, $0 \leqslant k \leqslant J$, is denoted n_k (hence $n_J = n$).

Our analysis applies to symmetric multigrid schemes based on the Galerkin principle for the SPD system (1.1); that is, restriction is the transpose of prolongation and the matrix A_k at level k, $k = J-1, \dots, 0$, is given by $A_k = P_k^T A_{k+1} P_k$, where P_k is the prolongation operator from level k to level $k+1$; we also assume that the smoother R_k is SPD and that the number of pre-smoothing steps ν ($\nu > 0$) is equal to the number of post-smoothing steps. The algorithm for V-cycle multigrid is then as follows.

Multigrid with V-cycle at level k: $x_{n+1} = \mathrm{MG}(b, A_k, x_n, k)$

(1) Relax ν times with smoother R_k: $\bar{x}_n = Smooth(x_n, A_k, R_k, \nu, b)$
(2) Compute residual: $r_k = b - A_k\bar{x}_n$
(3) Restrict residual: $r_{k-1} = P_{k-1}^T r_k$
(4) Coarse grid correction: **if** $k = 1$, $e_0 = A_0^{-1} r_0$
else $e_{k-1} = \mathrm{MG}(r_{k-1}, A_{k-1}, 0, k-1)$
(5) Prolongate coarse grid correction: $\hat{\bar{x}}_n = \bar{x}_n + P_{k-1}e_{k-1}$
(6) Relax ν times with smoother R_k: $x_{n+1} = Smooth(\hat{\bar{x}}_n, A_k, R_k, \nu, b)$

When applying this algorithm, the error satisfies

$$A_k^{-1}b - x_{n+1} = E_{MG}^{(k)}\bigl(A_k^{-1}b - x_n\bigr)$$

where the iteration matrix $E_{MG}^{(k)}$ is recursively defined from

$$\begin{aligned} &E_{MG}^{(0)} = 0 \quad \text{and,} \quad \text{for } k = 1, 2, \ldots, J: \\ &E_{MG}^{(k)} = \bigl(I - R_k^{-1}A_k\bigr)^{\nu}\bigl(I - P_{k-1}\bigl(I - E_{MG}^{(k-1)}\bigr)A_{k-1}^{-1}P_{k-1}^T A_k\bigr)\bigl(I - R_k^{-1}A_k\bigr)^{\nu} \end{aligned} \tag{2.1}$$

(see, e.g., [16, p. 48]). Our main objective is the analysis of the spectral radius of $E_{MG}^{(J)}$, which governs convergence on the finest level. Our analysis makes use of the following general assumptions.

General assumptions

- $n = n_J > n_{J-1} > \cdots > n_0$;
- P_k is an $n_{k+1} \times n_k$ matrix of rank n_k, $k = J-1, \ldots, 0$;
- $A_J = A$ and $A_k = P_k^T A_{k+1} P_k$, $k = J-1, \ldots, 0$;
- R_k is SPD and such that $\rho(I - R_k^{-1}A_k) < 1$, $k = J, \ldots, 1$.

Note also that most of our results do not refer explicitly to the smoother R_k, but are stated with respect to the matrices $M_k^{(\nu)}$ defined from

$$I - M_k^{(\nu)^{-1}}A_k = \bigl(I - R_k^{-1}A_k\bigr)^{\nu}. \tag{2.2}$$

That is, $M_k^{(\nu)}$ is the smoother that provides in 1 step the same effect as ν steps with R_k. The results stated with respect to $M_k^{(\nu)}$ may then be seen as results stated for the case of 1 pre- and 1 post-smoothing step, which can be extended to the general case via the relations (2.2).

Most results depend on the following parameter:

$$\omega^{(\nu)} = \max\left(1, \max_{1\leqslant k\leqslant J} \max_{w_k\in\mathbb{R}^{n_k}} \frac{w_k^T A_k w_k}{w_k^T M_k^{(\nu)} w_k}\right). \tag{2.3}$$

From $\rho(I - R_k^{-1}A_k) < 1$, it follows that $\omega^{(1)} < 2$, whereas (2.2) implies

$$\omega^{(\nu)} = \begin{cases} 1 & \text{if } \nu \text{ is even,} \\ 1 + (\omega^{(1)} - 1)^{\nu} & \text{if } \nu \text{ is odd.} \end{cases} \tag{2.4}$$

Hence one has also $\omega^{(\nu)} < 2$ for all ν. Further, if $\omega^{(1)} = 1$, then $\omega^{(\nu)} = 1$ for all ν.

We close this subsection by introducing the projector π_{A_k} which plays an important role throughout this paper:

$$\pi_{A_k} = P_{k-1}A_{k-1}^{-1}P_{k-1}^T A_k. \tag{2.5}$$

3. Bounds on the V-cycle multigrid convergence factor

3.1. SSC theory

We consider the SSC convergence analysis as presented in Theorem 4.4 and Lemma 4.6 in [18], and Theorem 5.1 in [20]. Of course, there are more recent versions of this theory, e.g., in [19] it is obtained an *identity* (known as *XZ-identity*) which provides the exact convergence factor. However, we do not see how to transform these further versions so that, according to the focus of this paper, they deliver a bound that could be assessed considering only two levels at a time (while being significantly different from the bound given by Theorem 3.1 together with Theorem 3.3). In particular, it seems clear that the exact convergence factor is a global quantity whose knowledge necessarily involves information from all levels. Note that SSC ideas are also treated in algebraic setting in [17, Section 5], where both the XZ-identity and approximation property approaches are presented, without however comparing them.

Now, we first develop in Theorem 3.1 below an algebraic version of Theorem 5.1 in [20]. We give a complete proof since this version slightly improves the original formulation, which uses a matrix Γ with same entries in the strict upper part, but nonnegative entries in the strict lower part and positive entries on the diagonal.

Observe that in Theorem 3.1 below the freedom left in choosing the pseudo restrictions G_k corresponds, in the original formulation, to the freedom associated with the choice of the space decomposition. More precisely, given a set of G_k, $k=0,\dots,J-1$, we can construct a corresponding space decomposition as defined in [20]. In Appendix A we show that the converse is also true; that is, with any admissible space decomposition in the original theory, one may associate a set of pseudo restrictions G_k such that Theorem 3.1 will yield the same bound as Theorem 5.1 in [20], except for the improvement associated with the refined definition of Γ.

Theorem 3.1. *Let $E_{MG}^{(J)}$ be defined by* (2.1) *with P_k, $k=0,\dots,J-1$, A_k, $k=0,\dots,J$, and R_k, $k=1,\dots,J$, satisfying the general assumptions stated in Section* 2. *For $k=1,\dots,J$, let $M_k^{(\nu)}$ be defined by* (2.2), *and set $M_0^{(\nu)}=A_0$.*

Let G_k, $k=0,\dots,J-1$, be $n_k\times n_{k+1}$ matrices, and, for $k=0,\dots,J$, let $\check{P}_k$ and $\check{G}_k$ be defined by, respectively,

$$\begin{aligned}&\check{P}_J=I,\\&\check{P}_k=\check{P}_{k+1}P_k,\quad k=J-1,\dots,0,\end{aligned}\tag{3.1}$$

and

$$\begin{aligned}&\check{G}_J=I,\\&\check{G}_k=G_k\check{G}_{k+1},\quad k=J-1,\dots,0,\end{aligned}\tag{3.2}$$

with $P_{-1}=G_{-1}=O$.

There holds

$$\rho\big(E_{MG}^{(J)}\big)\leqslant 1-\frac{2-\omega^{(\nu)}}{K^{(\nu)}(1+\|\Gamma\|)^2},\tag{3.3}$$

where $\omega^{(\nu)}$ is defined by (2.3),

$$K^{(\nu)}=\max_{v\in\mathbb{R}^n}\frac{\sum_{k=0}^J v^T\check{G}_k^T(I-P_{k-1}G_{k-1})^T M_k^{(\nu)}(I-P_{k-1}G_{k-1})\check{G}_k v}{v^TAv},\tag{3.4}$$

and

$$\Gamma=\begin{pmatrix}0&\gamma_{01}&\cdots&\gamma_{0J}\\&0&\cdots&\gamma_{1J}\\&&\ddots&\vdots\\&&0&\gamma_{(J-1)J}\\&&&0\end{pmatrix},\tag{3.5}$$

with, for $k=0,\dots,J-1$ and $l=k+1,\dots,J$,

$$\gamma_{kl}=\max_{w_k\in\mathbb{R}^{n_k}}\max_{v\in\mathbb{R}^n}\frac{v^T\check{G}_l^T(I-P_{l-1}G_{l-1})^T\check{P}_l^TA\check{P}_kw_k}{(w_k^TM_k^{(\nu)}w_k)^{1/2}(v^T\check{G}_l^T(I-P_{l-1}G_{l-1})^TM_l^{(\nu)}(I-P_{l-1}G_{l-1})\check{G}_lv)^{1/2}}.\tag{3.6}$$

Moreover,

$$\|\Gamma\|\leqslant\omega^{(\nu)}\sqrt{J(J+1)/2}.\tag{3.7}$$

Proof. In what follows, we omit the superscript (ν) in $M_k^{(\nu)}$. We first gather some useful definitions:

$$Q_k=(I-P_{k-1}G_{k-1})\check{G}_k,\qquad k=0,\dots,J;\tag{3.8}$$

$$T_k=\check{P}_k(M_k)^{-1}\check{P}_k^TA,\qquad k=0,\dots,J;\tag{3.9}$$

$$F_k=(I-T_k)(I-T_{k-1})\cdots(I-T_1)(I-T_0),\quad k=0,\dots,J.\tag{3.10}$$

In addition we set $F_{-1}=I$,

As shown in [17, Proposition 5.1.1] there holds

$$E_{MG}^{(J)}=(I-T_J)(I-T_{J-1})\cdots(I-T_1)(I-T_0)(I-T_1)\cdots(I-T_{J-1})(I-T_J).$$

Further, since $A^{-1}(I-T_k)^T=(I-T_k)A^{-1}$ and $(I-T_0)^2=I-T_0$, one has $E_{MG}^{(J)}=F_J A^{-1} F_J^T A$, showing that

$$\rho\bigl(E_{MG}^{(J)}\bigr)=\|F_J\|_A^2=\max_{v\in\mathbb{R}^n}\frac{\|F_J v\|_A^2}{v^T A v}. \tag{3.11}$$

Using this relation, we first show that (3.3) holds if

$$v^T A v\leqslant K\bigl(1+\|\Gamma\|\bigr)^2\left(\sum_{l=0}^{J} v^T F_{l-1}^T A T_l F_{l-1} v\right)\quad \forall v\in\mathbb{R}^n. \tag{3.12}$$

Indeed, since $AT_k=T_k^T A$ and using (2.3), one has, $\forall v\in\mathbb{R}^n$,

$$\begin{aligned}
\|F_{k-1}v\|_A^2-\|F_k v\|_A^2 &= (F_{k-1}v)^T A F_{k-1}v-(F_{k-1}v)^T (I-T_k)^T A(I-T_k)F_{k-1}v\\
&= 2v^T F_{k-1}^T A T_k F_{k-1}v-(F_{k-1}v)^T T_k^T A T_k (F_{k-1}v)\\
&= 2v^T F_{k-1}^T A T_k F_{k-1}v-(F_{k-1}v)^T A\check{P}_k M_k^{-1}\check{P}_k^T A\check{P}_k M_k^{-1}\check{P}_k^T A(F_{k-1}v)\\
&= 2v^T F_{k-1}^T A T_k F_{k-1}v-(F_{k-1}v)^T A\check{P}_k M_k^{-1} A_k M_k^{-1}\check{P}_k^T A(F_{k-1}v)\\
&\geqslant 2v^T F_{k-1}^T A T_k F_{k-1}v-\omega^{(\nu)}(F_{k-1}v)^T A\check{P}_k M_k^{-1}\check{P}_k^T A(F_{k-1}v)\\
&= \bigl(2-\omega^{(\nu)}\bigr)v^T F_{k-1}^T A T_k F_{k-1}v.
\end{aligned}$$

Summing both sides for $k=0,\dots,J$ shows that, $\forall v\in\mathbb{R}^n$,

$$\|v\|_A^2-\|F_J v\|_A^2\geqslant\bigl(2-\omega^{(\nu)}\bigr)\left(\sum_{l=0}^{J} v^T F_{l-1}^T A T_l F_{l-1} v\right),$$

and it is straightforward to check that this relation, together with (3.12) and (3.11), implies (3.3).

We now prove (3.12). Observe that, using (3.8), there holds

$$\sum_{l=0}^{J}\check{P}_l Q_l=\sum_{l=0}^{J}\check{P}_l(I-P_{l-1}G_{l-1})\check{G}_l=\sum_{l=0}^{J}\bigl(\check{P}_l\check{G}_l-\check{P}_{l-1}\check{G}_{l-1}\bigr)=\check{P}_J\check{G}_J-\check{P}_{-1}\check{G}_{-1}=I.$$

For any $v\in\mathbb{R}^n$, one may then decompose $v^T A v$ as the sum of two terms (remembering that $F_{-1}=I$):

$$v^T A v=\sum_{l=0}^{J} v^T A\check{P}_l Q_l v=\sum_{l=0}^{J} v^T F_{l-1}^T A\check{P}_l Q_l v+\sum_{l=1}^{J} v^T\bigl(I-F_{l-1}^T\bigr)A\check{P}_l Q_l v. \tag{3.13}$$

In order to prove (3.12), we bound separately the two terms in the right-hand side of (3.13).

Regarding the first term, one has, applying twice Cauchy–Schwartz inequality,

$$\begin{aligned}
\sum_{l=0}^{J} v^T F_{l-1}^T A\check{P}_l Q_l v &\leqslant \sum_{l=0}^{J}\bigl(v^T Q_l^T M_l Q_l v\bigr)^{1/2}\bigl(v^T F_{l-1}^T A\check{P}_l M_l^{-1}\check{P}_l^T A F_{l-1}v\bigr)^{1/2}\\
&\leqslant \left(\sum_{l=0}^{J} v^T Q_l^T M_l Q_l v\right)^{1/2}\left(\sum_{l=0}^{J} v^T F_{l-1}^T A T_l F_{l-1} v\right)^{1/2}.
\end{aligned} \tag{3.14}$$

To estimate the second term, first observe that

$$I-F_{l-1}=I-(I-T_{l-1})F_{l-2}=(I-F_{l-2})+T_{l-1}F_{l-2}=\cdots=\sum_{k=0}^{l-1}T_k F_{k-1}.$$

Therefore,

$$\sum_{l=1}^{J} v^T\bigl(I-F_{l-1}^T\bigr)A\check{P}_l Q_l v=\sum_{l=1}^{J}\sum_{k=0}^{l-1} v^T F_{k-1}^T T_k^T A\check{P}_l Q_l v,$$

whereas, for any $0\leqslant k<l\leqslant J$, using successively (3.9) and (3.6) with $w_k=M_k^{-1}\check{P}_k^T A F_{k-1}v$,

$$
\begin{aligned}
v^T F_{k-1}^T T_k^T A \check{P}_l Q_l v &= \left(v^T F_{k-1}^T A \check{P}_k M_k^{-1}\right) \check{P}_k^T A \check{P}_l Q_l v \\
&\leqslant \gamma_{kl} \left(v^T Q_l^T M_l Q_l v\right)^{1/2} \left(v^T F_{k-1}^T A \check{P}_k M_k^{-1} \check{P}_k^T A F_{k-1} v\right)^{1/2} \\
&= \gamma_{kl} \left(v^T Q_l^T M_l Q_l v\right)^{1/2} \left(v^T F_{k-1}^T A T_k F_{k-1} v\right)^{1/2}.
\end{aligned}
$$

Hence, since $\|\Gamma\| = \max_y \frac{\|\Gamma y\|}{\|y\|} = \max_{x,y} \frac{x^T \Gamma y}{\|x\|\|y\|}$ and using the definition (3.5) of Γ, there holds

$$
\begin{aligned}
\sum_{l=1}^{J} v^T \left(I - F_{l-1}^T\right) A \check{P}_l Q_l v &\leqslant \sum_{l=1}^{J} \sum_{k=0}^{l-1} \gamma_{kl} \left(v^T Q_l^T M_l Q_l v\right)^{1/2} \left(v^T F_{k-1}^T A T_k F_{k-1} v\right)^{1/2} \\
&\leqslant \|\Gamma\| \left(\sum_{l=0}^{J} v^T Q_l^T M_l Q_l v \right)^{1/2} \left(\sum_{k=0}^{J} v^T F_{k-1}^T A T_k F_{k-1} v \right)^{1/2}.
\end{aligned}
$$

Combining this latter result with (3.14), one gets

$$
v^T A v \leqslant (1 + \|\Gamma\|) \left(\sum_{l=0}^{J} v^T Q_l^T M_l Q_l v \right)^{1/2} \left(\sum_{l=0}^{J} v^T F_{l-1}^T A T_l F_{l-1} v \right)^{1/2}.
$$

Taking the square of both sides, and using (3.4) (which amounts to $\sum_{l=0}^{J} v^T Q_l^T M_l Q_l v \leqslant K v^T A v$) straightforwardly leads to (3.12), which completes the proof of (3.3).

It remains to prove (3.7). Note that $\|\Gamma\| \leqslant \|\Gamma\|_{\mathcal{F}} = (\sum_{l=1}^{J} \sum_{k=0}^{l-1} \gamma_{kl}^2)^{1/2}$. Further, for any $0 \leqslant k < l \leqslant J$ and for any $w \in \mathbb{R}^n$ and $w_k \in \mathbb{R}^{n_k}$,

$$
\begin{aligned}
w^T Q_l^T \check{P}_l^T A \check{P}_k w_k &\leqslant \left(w^T Q_l^T \check{P}_l^T A \check{P}_l Q_l w\right)^{1/2} \left(w_k^T \check{P}_k^T A \check{P}_k w_k\right)^{1/2} \\
&= \left(w^T Q_l^T A_l Q_l w\right)^{1/2} \left(w_k^T A_k w_k\right)^{1/2} \\
&\leqslant \omega^{(\nu)} \left(w^T Q_l^T M_l Q_l w\right)^{1/2} \left(w_k^T M_k w_k\right)^{1/2},
\end{aligned}
$$

showing that $\gamma_{kl} \leqslant \omega^{(\nu)}$. The required result straightforwardly follows. □

Now, in this paper, we focus on bounds that can be estimated considering only two consecutive levels at a time. The following theorem helps to see when the main constant $K^{(\nu)}$ in Theorem 3.1 can be set in that form.

Theorem 3.2. *Let $\check{P}_k$ and $\check{G}_k$ be defined by* (3.1) *and* (3.2) *with P_k, $k = 0, \ldots, J-1$, and A_k, $k = 0, \ldots, J$, satisfying the general assumptions stated in Section* 2. *Then, for all $v \in \mathbb{R}^n$*

$$
\begin{aligned}
v^T A v &= \sum_{k=0}^{J} v^T \check{G}_k^T (I - P_{k-1} G_{k-1})^T A_k (I - P_{k-1} G_{k-1}) \check{G}_k v \\
&\quad + 2 \sum_{k=0}^{J} v^T \check{G}_{k-1}^T P_{k-1}^T A_k (I - P_{k-1} G_{k-1}) \check{G}_k v \qquad (3.15) \\
&= \sum_{k=0}^{J} v^T \check{G}_k^T (I - P_{k-1} G_{k-1})^T A_k (I + P_{k-1} G_{k-1}) \check{G}_k v. \qquad (3.16)
\end{aligned}
$$

Moreover, if $P_{k-1} G_{k-1}$ is a projector, then

$$
(I - P_{k-1} G_{k-1})^T A_k (I + P_{k-1} G_{k-1}) \qquad (3.17)
$$

is nonnegative definite if and only if

$$
G_{k-1} = A_{k-1}^{-1} P_{k-1}^T A_k. \qquad (3.18)
$$

Proof. We begin, noting that $v_k^T A_k P_{k-1} G_{k-1} v_k = (v_k^T A_k P_{k-1} G_{k-1} v_k)^T = v_k^T (P_{k-1} G_{k-1})^T A_k v_k$ holds for all $v_k \in \mathbb{R}^{n_k}$. Using this relation with $v_k = \check{G}_k v$, Eqs. (3.15) and (3.16) follow from

$$\sum_{k=0}^{J} v^T \check{G}_k^T (I + P_{k-1}G_{k-1})^T A_k (I - P_{k-1}G_{k-1}) \check{G}_k v$$
$$= \sum_{k=0}^{J} v^T \check{G}_k^T \check{P}_k^T A \check{P}_k \check{G}_k v - v^T \check{G}_{k-1}^T \check{P}_{k-1}^T A \check{P}_{k-1} \check{G}_{k-1} v = v^T A v.$$

Next, $(I - P_{k-1}G_{k-1})^T A_k (I + P_{k-1}G_{k-1})$ is nonnegative definite if and only if

$$v_k^T (I - P_{k-1}G_{k-1})^T A_k (I + P_{k-1}G_{k-1}) v_k \geqslant 0 \quad \forall v_k \in \mathbb{R}^{n_k}$$

which in turn is equivalent to

$$v_k^T A_k v_k \geqslant v_k^T (P_{k-1}G_{k-1})^T A_k P_{k-1} G_{k-1} v_k \quad \forall v_k \in \mathbb{R}^{n_k},$$

this latter being nothing else but

$$\| P_{k-1} G_{k-1} \|_{A_k} \leqslant 1.$$

Hence, if $P_{k-1}G_{k-1}$ is a projector, it has to be orthogonal, and, hence, symmetric with respect to the $(\cdot, A_k \cdot)$ inner product (see [12, Section 5.13]); that is, $P_{k-1}G_{k-1} = B_k A_k$ for some symmetric B_k. This implies $G_{k-1} = C_{k-1} P_{k-1}^T A_k$ with C_{k-1} symmetric. Since P_{k-1} has full rank, $P_{k-1}G_{k-1}$ is then a projector if and only if $C_{k-1} = A_{k-1}$; hence the required result. □

Now, consider the definition (3.4) of $K^{(\nu)}$. To obtain an expression that can be assessed considering only two levels at a time, the only possibility we have found is to express the denominator $v^T A v$ as a sum over all levels similar to the sum in the numerator, and, assuming each term involved nonnegative, to bound the ratio of both these sums $\sum_k a_k / \sum_k b_k$ by the maximum of the ratios $\max_k (a_k / b_k)$. The first result of Theorem 3.2 tells us that such a splitting of $v^T A v$ always exists, but the second result tells us that it is exploitable only with $G_{k-1} = A_{k-1}^{-1} P_{k-1}^T A_k$, since otherwise there would be negative terms in the sum of the denominator, at least for certain v.[3] Note that these G_k are such that $P_{k-1}G_{k-1} = \pi_{A_k}$ and correspond to the so-called a-orthogonal decomposition in the original abstract theory. This choice is further analyzed in the following theorem, where we prove in particular that one has then $\Gamma = 0$. Note that with the original formulation of [20, Theorem 5.1], one could only prove $\|\Gamma\| \leqslant \omega^{(\nu)}$.

Theorem 3.3. *Let the assumptions of Theorem* 3.1 *hold, and let G_k, $k = 0, \ldots, J-1$, be defined by* (3.18)*. Then, $K^{(\nu)}$ and Γ, defined as in Theorem* 3.1*, satisfy, respectively*

$$K^{(\nu)} = \max\left(1, \max_{1 \leqslant k \leqslant J} \max_{w_k \in \mathbb{R}^{n_k}} \frac{w_k^T (I - \pi_{A_k})^T M_k^{(\nu)} (I - \pi_{A_k}) w_k}{w_k^T (I - \pi_{A_k})^T A_k (I - \pi_{A_k}) w_k}\right) \tag{3.19}$$

$$= \max\left(1, \max_{1 \leqslant k \leqslant J} \max_{w_k \in \mathbb{R}^{n_k}} \frac{w_k^T (I - \pi_{A_k})^T M_k^{(\nu)} (I - \pi_{A_k}) w_k}{w_k^T A_k w_k}\right) \tag{3.20}$$

and

$$\Gamma = 0, \tag{3.21}$$

where π_{A_k} is defined by (2.5)*.*

Proof. We first prove (3.21). Note that (3.18) implies $\check{G}_l = A_l^{-1} \check{P}_l^T A$, $l = 0, \ldots, J-1$. Hence, for any $0 \leqslant k < l \leqslant J$ and all $w_k \in \mathbb{R}^{n_k}$, $v \in \mathbb{R}^n$,

$$\begin{aligned}
w_k^T \check{P}_k^T A \check{P}_l (I - P_{l-1}G_{l-1}) \check{G}_l v &= w_k^T \check{P}_k^T A \check{P}_l A_l^{-1} \check{P}_l^T A v - w_k^T \check{P}_k^T A \check{P}_{l-1} A_{l-1}^{-1} \check{P}_{l-1}^T A v \\
&= w_k^T P_k^T \cdots P_{l-1}^T \big(\check{P}_l^T A \check{P}_l A_l^{-1} \big) \check{P}_l^T A v \\
&\quad - w_k^T P_k^T \cdots P_{l-2}^T \big(\check{P}_{l-1}^T A \check{P}_{l-1} A_{l-1}^{-1} \big) \check{P}_{l-1}^T A v \\
&= w_k^T P_k^T \cdots P_{l-1}^T \check{P}_l^T A v - w_k^T P_k^T \cdots P_{l-2}^T \check{P}_{l-1}^T A v \\
&= w_k^T \check{P}_k^T A v - w_k^T \check{P}_k^T A v \\
&= 0;
\end{aligned}$$

$\gamma_{kl} = 0$ and therefore $\Gamma = 0$ readily follows.

[3] Theorem 3.2 proves this under the additional assumption that $P_k G_k$ is a projector, but we did not found any usable bound based on G_k for which $P_k G_k$ would not be a projector.

We next prove (3.19) and (3.20). Using (3.16) and $P_{k-1}G_{k-1} = \pi_{A_k}$ together with $(I + \pi_{A_k})^T A_k (I - \pi_{A_k}) = (I - \pi_{A_k})^T \times A_k(I - \pi_{A_k})$ in the definition (3.4) of $K^{(\nu)}$, one has

$$K^{(\nu)} = \max_{v\in\mathbb{R}^n} \frac{\sum_{k=0}^{J} v^T \check{G}_k^T (I - P_{k-1}G_{k-1})^T M_k^{(\nu)} (I - P_{k-1}G_{k-1})\check{G}_k v}{\sum_{k=0}^{J} v^T \check{G}_k^T (I - P_{k-1}G_{k-1})^T A_k (I - P_{k-1}G_{k-1})\check{G}_k v}$$
$$= \max_{v\in\mathbb{R}^n} \frac{\sum_{k=1}^{J} v^T \check{G}_k^T (I - \pi_{A_k})^T M_k^{(\nu)} (I - \pi_{A_k})\check{G}_k v + v^T \check{G}_0^T A_0 \check{G}_0 v}{\sum_{k=1}^{J} v^T \check{G}_k^T (I - \pi_{A_k})^T A_k (I - \pi_{A_k})\check{G}_k v + v^T \check{G}_0^T A_0 \check{G}_0 v}$$
$$\leqslant \max\left(1, \max_{1\leqslant k\leqslant J} \max_{w_k\in\mathbb{R}^{n_k}} \frac{w_k^T (I - \pi_{A_k})^T M_k^{(\nu)} (I - \pi_{A_k}) w_k}{w_k^T (I - \pi_{A_k})^T A_k (I - \pi_{A_k}) w_k}\right). \tag{3.22}$$

This proves that the right-hand side of (3.19) is an upper bound on $K^{(\nu)}$; the right-hand side of (3.20) is a further upper bound since

$$\max_{w_k\in\mathbb{R}^{n_k}} \frac{w_k^T (I - \pi_{A_k})^T M_k^{(\nu)} (I - \pi_{A_k}) w_k}{w_k^T A_k w_k} \geqslant \max_{v_k\in\mathbb{R}^{n_k}} \frac{v_k^T (I - \pi_{A_k})^T M_k^{(\nu)} (I - \pi_{A_k}) v_k}{v_k^T (I - \pi_{A_k})^T A_k (I - \pi_{A_k}) v_k},$$

as seen by restricting the maximum in the left-hand side to $w_k = (I - \pi_{A_k})v_k$ (taking into account that $(I - \pi_{A_k})^2 = (I - \pi_{A_k})$).

To prove that the right-hand sides of (3.19), (3.20) are also a lower bound on $K^{(\nu)}$, let, for $k = 0, \ldots, J$, $\check{Q}_k = (I - P_{k-1}G_{k-1})\check{G}_k$. Then rewrite (3.22) as

$$K^{(\nu)} = \max_{v\in\mathbb{R}^n} \frac{\sum_{k=0}^{J} v^T \check{Q}_k^T M_k^{(\nu)} \check{Q}_k v}{\sum_{k=0}^{J} v^T \check{Q}_k^T A_k \check{Q}_k v}. \tag{3.23}$$

Since $G_k P_k = I_{n_k}$ for $k = 0, \ldots, J-1$, Lemma B.1 in Appendix B proves that, for $0 \leqslant l, k \leqslant J$ with $k \neq l$,

$$\check{Q}_l \check{P}_l \check{Q}_l = \check{Q}_l \quad \text{and} \quad \check{Q}_k \check{P}_l \check{Q}_l = O_{n_k\times n}.$$

Restricting the maximum in (3.23) to $v = \check{P}_l \check{Q}_l w$ for some $0 \leqslant l \leqslant J$ yields

$$K^{(\nu)} \geqslant \max_{w\in\mathbb{R}^n} \frac{w^T \check{Q}_l^T M_l^{(\nu)} \check{Q}_l w}{w^T \check{Q}_l^T A_l \check{Q}_l w}$$
$$= \max_{w\in\mathbb{R}^n} \frac{w^T \check{G}_l^T (I - P_{l-1}G_{l-1})^T M_l^{(\nu)} (I - P_{l-1}G_{l-1})\check{G}_l w}{w^T \check{G}_l^T (I - P_{l-1}G_{l-1})^T A_l (I - P_{l-1}G_{l-1})\check{G}_l w}$$
$$= \max_{w_l\in\mathbb{R}^{n_l}} \frac{w_l^T (I - P_{l-1}G_{l-1})^T M_l^{(\nu)} (I - P_{l-1}G_{l-1}) w_l}{w_l^T (I - P_{l-1}G_{l-1})^T A_l (I - P_{l-1}G_{l-1}) w_l},$$

the last equality stemming from the fact that G_l, and hence $\check{G}_l$, has full rank (from (3.18), (3.2), and because P_k has full rank by virtue of our general assumptions). The conclusion follows because

$$w_l^T (I - P_{l-1}G_{l-1})^T A_l (I - P_{l-1}G_{l-1}) w_l = w_l^T (I - \pi_{A_l})^T A_l (I - \pi_{A_l}) w_l$$
$$= w_l^T \left(A_l - A_l P_{l-1} A_{l-1}^{-1} P_{l-1}^T A_l\right) w_l$$
$$\leqslant w_l^T A_l w_l. \qquad \square$$

3.2. Hackbusch bound

The bound from [8, Theorem 7.2.2] is recalled in the following theorem. Note that this analysis requires $\omega^{(\nu)} = 1$. This condition is however not too restrictive since the smoother can be scaled to satisfy it. Note also that, according to (2.4), $\omega^{(\nu)} = 1$ always holds for ν even, and that $\omega^{(1)} = 1$ entails $\omega^{(\nu)} = 1$ for all ν.

Theorem 3.4. *Let $E_{MG}^{(J)}$ be defined by* (2.1) *with P_k, $k = 0, \ldots, J-1$, A_k, $k = 0, \ldots, J$, and R_k, $k = 1, \ldots, J$, satisfying the general assumptions stated in Section* 2. *For $k = 1, \ldots, J$, let $M_k^{(\nu)}$ and $\omega^{(\nu)}$ be defined, respectively, by* (2.2) *and* (2.3).

Then, if $\omega^{(\nu)} = 1$,

$$\rho\left(E_{MG}^{(J)}\right) \leqslant \frac{c_A^{(\nu)}}{c_A^{(\nu)} + 2}, \tag{3.24}$$

where

$$c_A^{(\nu)} = \max_{1\leqslant k\leqslant J} \max_{v_k\in\mathbb{R}^{n_k}} \frac{v_k^T(A_k^{-1} - P_{k-1}A_{k-1}^{-1}P_{k-1}^T)v_k}{v_k^T M_k^{(\nu)^{-1}} v_k}. \tag{3.25}$$

Moreover, if $\omega^{(1)} = 1$,

$$\rho\big(E_{MG}^{(J)}\big) \leqslant \frac{c_A^{(1)}}{c_A^{(1)} + 2\nu}. \tag{3.26}$$

Note that Theorem 7.2.2 in [8] considers only (3.26). The bound (3.24) is a straightforward extension (through the replacement of $M_k^{(1)} = R_k$ by $M_k^{(\nu)}$) that will make easier the comparison with other approaches. It is not really useful in practice since, as will be seen, (3.26) is always better than (3.24). Note, however, that (3.24) is more general since one may have $\omega^{(\nu)} = 1$ while $\omega^{(1)} > 1$.

Note also that in [8] some bounds based on c_A are also proved for the W- and two-grid cycle, that are better than those obtained by using just the V-cycle bound as a worst case estimate.

3.3. McCormick's bound

We recall in the following theorem the bound obtained in [11, Lemma 2.3, Theorem 3.4 and Section 5] (see also [10], or [15] for an alternative proof).

Theorem 3.5. *Let $E_{MG}^{(J)}$ be defined by* (2.1) *with P_k, $k = 0, \ldots, J-1$, A_k, $k = 0, \ldots, J$, and R_k, $k = 1, \ldots, J$, satisfying the general assumptions stated in Section* 2. *For $k = 1, \ldots, J$, let $M_k^{(\nu)}$ be defined by* (2.2).

Then,

$$\rho\big(E_{MG}^{(J)}\big) \leqslant 1 - \delta^{(\nu)}, \tag{3.27}$$

where

$$\delta^{(\nu)} = \min_{1\leqslant k\leqslant J} \min_{v_k\in\mathbb{R}^{n_k}} \frac{\|v_k\|_{A_k}^2 - \|(I - M_k^{(\nu)^{-1}}A_k)v_k\|_{A_k}^2}{\|(I - \pi_{A_k})v_k\|_{A_k}^2} \tag{3.28}$$

with π_{A_k} defined by (2.5).

Moreover,

$$\delta^{(\nu)^{-1}} \leqslant \frac{1}{\nu}\big(\delta^{(1)^{-1}} + \nu - 1\big). \tag{3.29}$$

4. Comparison

We first state our main result, which relates the constants $K^{(\nu)}$, $c_A^{(\nu)}$ and $\delta^{(\nu)}$.

Theorem 4.1. *Let $K^{(\nu)}$, $c_A^{(\nu)}$ and $\delta^{(\nu)}$ be defined respectively by* (3.19), (3.25) *and* (3.28) *where P_k, $k = 0, \ldots, J-1$, A_k, $k = 0, \ldots, J$, and R_k, $k = 1, \ldots, J$, satisfy the general assumptions stated in Section* 2. *For $k = 1, \ldots, J$, let $M_k^{(\nu)}$ be defined by* (2.2).

Then

$$K^{(\nu)} = \max\big(1, c_A^{(\nu)}\big), \tag{4.1}$$

and

$$\delta^{(\nu)} = \frac{1}{c_A^{(2\nu)}}. \tag{4.2}$$

Proof. Let

$$\widetilde{P}_k = A_k^{1/2} P_{k-1} A_{k-1}^{-1/2}, \quad k = 1, \ldots, J.$$

One has

$$\begin{aligned}
c_A^{(\nu)} &= \max_{1\leqslant k\leqslant J}\max_{v\in\mathbb{R}^k}\frac{v^T(A_k^{-1}-P_{k-1}A_{k-1}^{-1}P_{k-1}^T)v}{v^T M_k^{(\nu)^{-1}}v}\\
&= \max_{1\leqslant k\leqslant J}\max_{v\in\mathbb{R}^k}\frac{v^T(I-A_k^{1/2}P_{k-1}A_{k-1}^{-1}P_{k-1}^TA_k^{1/2})v}{v^TA_k^{1/2}M_k^{(\nu)^{-1}}A_k^{1/2}v}\\
&= \max_{1\leqslant k\leqslant J}\max_{v\in\mathbb{R}^k}\frac{v^T(I-\widetilde{P}_k\widetilde{P}_k^T)v}{v^TA_k^{1/2}M_k^{(\nu)^{-1}}A_k^{1/2}v}\\
&= \max_{1\leqslant k\leqslant J}\max_{v\in\mathbb{R}^k}\frac{v^TM_k^{(\nu)^{1/2}}A_k^{-1/2}(I-\widetilde{P}_k\widetilde{P}_k^T)^2A_k^{-1/2}M_k^{(\nu)^{1/2}}v}{v^Tv}\\
&= \max_{1\leqslant k\leqslant J}\max_{v\in\mathbb{R}^k}\frac{v^T(I-\widetilde{P}_k\widetilde{P}_k^T)A_k^{-1/2}M_k^{(\nu)}A_k^{-1/2}(I-\widetilde{P}_k\widetilde{P}_k^T)v}{v^Tv}.
\end{aligned}$$

Since $(I-\widetilde{P}_k\widetilde{P}_k^T)A_k^{-1/2}=(I-A_k^{1/2}P_{k-1}A_{k-1}^{-1}P_{k-1}^TA_k^{1/2})A_k^{-1/2}=A_k^{-1/2}(I-\pi_{A_k})^T$, this leads to

$$c_A^{(\nu)} = \max_{1\leqslant k\leqslant J}\max_{v\in\mathbb{R}^k}\frac{v^T(I-\pi_{A_k})^TM_k^{(\nu)}(I-\pi_{A_k})v}{v^TA_kv},$$

hence (4.1).

On the other hand, observing that $M_k^{(2\nu)}$ satisfy

$$I-M_k^{(2\nu)^{-1}}A_k=\bigl(I-M_k^{(\nu)^{-1}}A_k\bigr)^2,\quad k=1,\ldots,J,$$

one has

$$\begin{aligned}
\delta^{(\nu)} &= \min_{1\leqslant k\leqslant J}\min_{v\in\mathbb{R}^k}\frac{\|v\|_{A_k}^2-\|I-M_k^{(\nu)^{-1}}A_kv\|_{A_k}^2}{\|(I-\pi_{A_k})v\|_{A_k}^2}\\
&= \min_{1\leqslant k\leqslant J}\min_{v\in\mathbb{R}^k}\frac{v^TA_kv-v^T(I-M_k^{(\nu)^{-1}}A_k)^TA_k(I-M_k^{(\nu)^{-1}}A_k)v}{v^T(I-\pi_{A_k})^TA_k(I-\pi_{A_k})v}\\
&= \min_{1\leqslant k\leqslant J}\min_{v\in\mathbb{R}^k}\frac{v^TA_kv-v^TA_k(I-M_k^{(\nu)^{-1}}A_k)^2v}{v^T(I-\pi_{A_k})^TA_k(I-\pi_{A_k})v}\\
&= \min_{1\leqslant k\leqslant J}\min_{v\in\mathbb{R}^k}\frac{v^TA_kv-v^TA_k(I-M_k^{(2\nu)^{-1}}A_k)v}{v^T(I-\pi_{A_k})^TA_k(I-\pi_{A_k})v}\\
&= \min_{1\leqslant k\leqslant J}\min_{v\in\mathbb{R}^k}\frac{v^TA_kM_k^{(2\nu)^{-1}}A_kv}{v^TA_k(I-\pi_{A_k})v}\\
&= \min_{1\leqslant k\leqslant J}\min_{v\in\mathbb{R}^k}\frac{v^TM_k^{(2\nu)^{-1}}v}{v^T(I-\pi_{A_k})A_k^{-1}v}\\
&= \frac{1}{c_A^{(2\nu)}}.\qquad\Box
\end{aligned}$$

We are now ready to compare the bounds (3.3), (3.24), (3.26) and (3.27). This is done in the following theorem.

Theorem 4.2. *Let $E_{MG}^{(J)}$ be defined by* (2.1) *with P_k, $k=0,\ldots,J-1$, A_k, $k=0,\ldots,J$, and R_k, $k=1,\ldots,J$, satisfying the general assumptions stated in Section* 2*. For $k=1,\ldots,J$, let $M_k^{(\nu)}$ and $\omega^{(\nu)}$ be defined, respectively, by* (2.2) *and* (2.3)*. Moreover, let $K^{(\nu)}$, $c_A^{(\nu)}$ and $\delta^{(\nu)}$ be defined respectively by* (3.19), (3.25) *and* (3.28)*.*

Then

$$\rho\bigl(E_{MG}^{(J)}\bigr)\leqslant 1-\delta^{(\nu)}\leqslant 1-\frac{2-\omega^{(\nu)}}{K^{(\nu)}}. \tag{4.3}$$

Further, if $\omega^{(\nu)} = 1$,

$$\rho\big(E_{MG}^{(J)}\big) \leqslant 1 - \delta^{(\nu)} \leqslant \frac{c_A^{(\nu)}}{c_A^{(\nu)} + 2}, \tag{4.4}$$

and, if $\omega^{(1)} = 1$,

$$\rho\big(E_{MG}^{(J)}\big) \leqslant 1 - \delta^{(\nu)} \leqslant \frac{c_A^{(1)}}{c_A^{(1)} + 2\nu} \leqslant \frac{c_A^{(\nu)}}{c_A^{(\nu)} + 2}. \tag{4.5}$$

Moreover,

$$1 - \frac{2 - \omega^{(\nu)}}{K^{(\nu)}} \leqslant 1 - \frac{2 - \omega^{(\nu)}}{2} \delta^{(\nu)}, \tag{4.6}$$

and, if $\omega^{(\nu)} = 1$,

$$\frac{c_A^{(\nu)}}{c_A^{(\nu)} + 2} \leqslant \frac{1}{\delta^{(\nu)} + 1} = 1 - \frac{\delta^{(\nu)}}{\delta^{(\nu)} + 1}. \tag{4.7}$$

Proof. Let us first prove two intermediate results:

$$\frac{c_A^{(\nu)}}{2} \leqslant c_A^{(2\nu)} \leqslant \frac{c_A^{(\nu)}}{2 - \omega^{(\nu)}} \tag{4.8}$$

and, if $\omega^{(\mu)} = 1$,

$$\frac{c_A^{(\mu)}}{\nu} \leqslant c_A^{(\mu\nu)} \leqslant \frac{1}{\nu}\big(c_A^{(\mu)} + \nu - 1\big), \quad \mu \in \mathbb{N}_0^+. \tag{4.9}$$

The first intermediate result (4.8) follows from

$$M_k^{(2\nu)} = M_k^{(\nu)}\big(2M_k^{(\nu)} - A_k\big)^{-1} M_k^{(\nu)}$$

combined with

$$2v_k^T M_k^{(\nu)} v_k \geqslant 2v_k^T M_k^{(\nu)} v_k - v_k^T A_k v_k \geqslant \big(2 - \omega^{(\nu)}\big) v_k^T M_k^{(\nu)} v_k, \quad \forall v_k \in \mathbb{R}^{n_k}.$$

We prove the second intermediate result (4.9) for $\mu = 1$; its generalization to $\mu > 1$ is performed replacing R_k by $M_k^{(\mu)}$ in the proof below. First, the right inequality (4.9) is a consequence of (3.29) since, using (4.2) one has

$$c_A^{(\nu)} = \delta^{(\nu/2)^{-1}} \leqslant \frac{1}{\nu}\big(\delta^{(1/2)^{-1}} + \nu - 1\big) = \frac{1}{\nu}\big(c_A^{(1)} + \nu - 1\big)$$

where $\delta^{(1/2)}$ corresponds to the V-cycle algorithm with a smoother $\widetilde{R}_k$ such that

$$I - R_k^{-1} A_k = \big(I - \widetilde{R}_k^{-1} A_k\big)^2.$$

Such $\widetilde{R}_k$ is indeed well defined since $\omega^{(1)} = 1$ entails that $I - A_k^{1/2} R_k^{-1} A_k^{1/2}$ is symmetric nonnegative definite. On the other hand, the left inequality (4.9) is a straightforward consequence of

$$v_k^T M_k^{(\nu)^{-1}} v_k \leqslant \nu v_k^T R_k^{-1} v_k, \quad \forall v_k \in \mathbb{R}^{n_k}$$

which we prove as follows. This relation holds if and only if

$$v_k^T A_k^{1/2} M_k^{(\nu)^{-1}} A_k^{1/2} v_k \leqslant \nu v_k^T A_k^{1/2} R_k^{-1} A_k^{1/2} v_k, \quad \forall v_k \in \mathbb{R}^{n_k}$$

which, in view of (2.2) and when $\omega^{(1)} = 1$, is satisfied if

$$1 - (1 - x)^\nu \leqslant \nu x \quad \forall x \in [0, 1];$$

that is, if, $\forall \lambda = 1 - x \in [0, 1)$,

$$\frac{1 - \lambda^\nu}{1 - \lambda} \leqslant \nu,$$

which is readily checked from $\frac{1-\lambda^\nu}{1-\lambda} = \sum_{i=0}^{\nu-1} \lambda^i < \nu$.

Now, the second inequality (4.3) follows from the right inequality (4.8) combined with (4.1) and (4.2). The second inequalities (4.4) and (4.5) are equivalent to, respectively

$$c_A^{(\nu)} c_A^{(2\nu)} \geqslant \bigl(c_A^{(\nu)} + 2\bigr)\bigl(c_A^{(2\nu)} - 1\bigr)$$

and

$$c_A^{(1)} c_A^{(2\nu)} \geqslant \bigl(c_A^{(1)} + 2\nu\bigr)\bigl(c_A^{(2\nu)} - 1\bigr).$$

These inequalities follow from the right inequality (4.9), used with $(\mu,\nu) = (\nu,2)$ and $(\mu,\nu) = (1,2\nu)$, respectively, combined with (4.2). Next, the last inequality of (4.5) is a consequence of the left inequality of (4.9) used with $(\mu,\nu) = (1,\nu)$. Finally, inequalities (4.6) and (4.7) follow from the left inequality (4.8) combined with (4.2) and (4.1), because $\delta^{(\nu)^{-1}} \geqslant 1$, as may be seen from

$$\begin{aligned}
\delta^{(\nu)^{-1}} &= c^{(2\nu)} \\
&= \max_{1\leqslant k\leqslant J} \max_{w_k \in \mathbb{R}^{n_k}} \frac{w_k^T (I-\pi_{A_k})^T M_k^{(2\nu)} (I-\pi_{A_k}) w_k}{w_k^T (I-\pi_{A_k})^T A_k (I-\pi_{A_k}) w_k} \\
&\geqslant \frac{1}{\omega^{(2\nu)}} \max_{1\leqslant k\leqslant J} \max_{w_k \in \mathbb{R}^{n_k}} \frac{w_k^T (I-\pi_{A_k})^T M_k^{(2\nu)} (I-\pi_{A_k}) w_k}{w_k^T (I-\pi_{A_k})^T M_k^{(2\nu)} (I-\pi_{A_k}) w_k} \\
&= 1. \qquad \square
\end{aligned}$$

From (4.3), (4.4) and (4.5), one sees that McCormick's bound is always the best one, whereas inequalities (4.6) and (4.7) show that all approaches are nevertheless qualitatively equivalent, since they give bounds which, at worst, correspond to McCormick's bound with main constant smaller by a modest factor.

5. Example

We consider the linear system resulting from the 9-point finite difference discretization of the two-dimensional Poisson problem

$$\begin{aligned}
&-\Delta u = f \quad \text{in } \Omega = (0,1)\times(0,1), \\
&u = 0 \quad \text{in } \partial\Omega
\end{aligned}$$

on a uniform grid of mesh size $h = 1/N_J$ in both directions. The matrix corresponds then, up to some scaling factor, to the following nine point stencil

$$\begin{bmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{bmatrix}. \tag{5.1}$$

We assume $N_J = 2^J N_0$ for some integer N_0, allowing J steps of regular geometric coarsening. We consider prolongations in form of the standard interpolation associated with bilinear finite element basis functions. The restriction P_k^T corresponds then to "full weighting", as defined in, e.g., [16].[4] With these choices, the stencil (5.1) is preserved throughout all grids (up to some unimportant scaling factor), and $c_A^{(\nu)}$ may be assessed by analyzing

$$\max_{w_k} \frac{w_k^T (I-\pi_{A_k})^T M_k^{(\nu)} (I-\pi_{A_k}) w_k}{w_k A_k w_k} \tag{5.2}$$

for a matrix A_k corresponding to stencil (5.1) applied on a grid with mesh size $h_k = 1/N_k$. Considering two successive grids is therefore sufficient, and, to alleviate notation, we let $N = N_k$, $A = A_k$, $M^{(\nu)} = M_k^{(\nu)}$, $P = P_{k-1}$, $A_c = A_{k-1} = P^T A P$ and $\pi_A = \pi_{A_k} = P A_c^{-1} P^T A$.

To assess (5.2), we resort to Fourier analysis. The eigenvectors of A are, for $m, l = 1, \ldots, N-1$, the functions

$$u_{m,l}^{(N)} = \sin(m\pi x)\sin(l\pi y)$$

evaluated at the grid points. The eigenvalue corresponding to $u_{m,l}^{(N)}$ is

$$\lambda_{m,l}^{(N)} = 4(3s_m + 3s_l - 4s_m s_l) \tag{5.3}$$

[4] Up to some scaling factor; the scalings of the prolongation and restriction are unimportant when using coarse grid matrices of the Galerkin type.

where

$$s_m = \sin^2(m\pi/2N), \qquad s_l = \sin^2(l\pi/2N). \tag{5.4}$$

The prolongation P satisfies (see, e.g., [16, p. 87])

$$P^T \begin{Bmatrix} u_{m,l}^{(N)} \\ u_{N-m,N-l}^{(N)} \\ -u_{N-m,l}^{(N)} \\ -u_{m,N-l}^{(N)} \end{Bmatrix} = 4 \begin{Bmatrix} (1-s_m)(1-s_l) \\ s_m s_l \\ s_m(1-s_l) \\ (1-s_m)s_l \end{Bmatrix} u_{m,l}^{(N/2)}$$

for $1 \leqslant m, l \leqslant N/2-1$, with $P^T u_{m,l}^{(N)} = 0$ for $m = N/2$ or $m = N/2$. Expressed in the Fourier basis (that is, in the basis of eigenvectors of A), $I - \pi_A$ is therefore block diagonal with, for $1 \leqslant m, l \leqslant N/2-1$, 4×4 blocks

$$(I-\pi_A)_{m,l} = I_4 - \bar{P}_{m,l}\big(\bar{A}_{m,l}^{(c)}\big)^{-1}\bar{P}_{m,l}^T \bar{A}_{m,l} \tag{5.5}$$

where

$$\bar{P}_{m,l}^T = 4\,(\,(1-s_m)(1-s_l) \quad s_m s_l \quad s_m(1-s_l) \quad (1-s_m)s_l\,),$$
$$\bar{A}_{m,l} = \operatorname{diag}\big(\lambda_{m,l}^{(N)}, \lambda_{N-m,N-l}^{(N)}, \lambda_{m,N-l}^{(N)}, \lambda_{N-m,l}^{(N)}\big),$$
$$\bar{A}_{m,l}^{(c)} = \bar{P}_{m,l}^T \bar{A}_{m,l} \bar{P}_{m,l} = 64\big(3s_m(1-s_m) + 3s_l(1-s_l) - 16 s_l(1-s_l)s_m(1-s_m)\big).$$

For $m = N/2$, $1 \leqslant l \leqslant N/2-1$ and $l = N/2$, $1 \leqslant m \leqslant N/2-1$, $(I-\pi_A)_{m,l} = I_2$ is a 2×2 identity block, whereas $(I-\pi_A)_{\frac{N}{2},\frac{N}{2}} = 1$ reduces to the scalar identity. If $M^{(\nu)}$ in the Fourier basis has the same block diagonal structure, we are left with the analysis of

$$\rho_{m,l} = \rho\big((I-\pi_A)_{m,l}^T \bar{M}_{m,l}^{(\nu)} (I-\pi_A)_{m,l} \bar{A}_{m,l}^{-1}\big). \tag{5.6}$$

Now, we consider more specifically damped Jacobi smoothing; that is $R_k = \omega_{\mathrm{Jac}}^{-1}\operatorname{diag}(A) = \omega_{\mathrm{Jac}}^{-1} 8I$, with $\omega_{\mathrm{Jac}} \in (0, 4/3)$ to ensure $\omega^{(1)} = (3/2)\omega_{\mathrm{Jac}} < 2$. Then, for any number of pre- and post-smoothing steps ν, $M^{(\nu)}$ is diagonal in the Fourier basis, with diagonal entries depending on the eigenvalues of A; that is (see (5.3)), depending on s_m and s_l. To obtain grid independent bounds, it is then interesting to consider $\rho_{m,l} = \rho(s_m, s_l)$ as a function of s_m, s_l, and to let these parameters vary continuously in $[0, 1]$, excluding the corner points where $s_m(1-s_m) = s_l(1-s_l) = 0$, which correspond to singularities. For all ν, $\rho(s_m, s_l)$ has the following symmetries: $\rho(s_l, s_m) = \rho(1-s_m, s_s) = \rho(s_m, 1-s_l) = \rho(1-s_m, 1-s_l)$. Further, numerical investigations reveal that the maximum on the considered domain is located at the boundary, i.e., corresponds to, e.g., $s_m = 0$. Because of the symmetries it is sufficient to analyze this latter case. One may check that $\rho(0, s_l)$ is the largest eigenvalue in modulus of

$$\frac{1}{4}\begin{pmatrix} \frac{s_l\mu_1 + s_l\mu_4}{3} & 0 & 0 & -\frac{s_l\mu_1 + s_l\mu_4}{3} \\ 0 & \frac{\mu_2}{3-(1-s_l)} & 0 & 0 \\ 0 & 0 & \frac{\mu_3}{3-s_l} & 0 \\ -\frac{\mu_1(1-s_l)+\mu_4(1-s_l)}{3} & 0 & 0 & \frac{(1-s_l)\mu_1 + (1-s_l)\mu_4}{3} \end{pmatrix},$$

where $\{\mu_i\}_{i=1,\dots,4}$ are the 4 diagonal entries of $M_{kl}^{(\nu)}$, given by

$$\mu_i = \frac{(A_{m,l})_{i,i}}{1 - (1 - \frac{\omega_{\mathrm{Jac}}}{2}(A_{m,l})_{i,i})^\nu}.$$

Thus

$$\rho(0, s_l) = \max\left(\frac{\mu_3}{3-s_l}, \frac{\mu_2}{3-(1-s_l)}, \frac{\mu_1+\mu_4}{3}\right),$$

and, injecting the expressions of μ_i,

$$\rho(0, s_l) = \max\left(\frac{1}{1-(1-\frac{\omega_{\mathrm{Jac}}}{2}(3-s_l))^{(\nu)}}, \frac{1}{1-(1-\frac{\omega_{\mathrm{Jac}}}{2}(2+s_l))^{\nu}}, \right.$$
$$\left. \frac{s_l}{1-(1-\frac{3\omega_{\mathrm{Jac}}}{2}s_l)^\nu} + \frac{1-s_l}{1-(1-\frac{3\omega_{\mathrm{Jac}}}{2}(1-s_l))^\nu}\right).$$

Table 1
Convergence factor of V-cycle (for $N_0 = 2$ and $J = 6$) and the corresponding bounds for $\nu = 1$; (*) the quantity exists, but does not correspond to the bound, since $\omega^{(1)} > 1$.

ω_{Jac}	$\omega^{(1)}$	$c_A^{(1)}$	$c_A^{(2)}$	$\frac{c_A^{(1)}}{c_A^{(1)}+2}$	$1-\frac{2-\omega^{(1)}}{K^{(1)}}$	$1-\delta^{(1)}$	$\rho(E_{MG}^{(J)})$
1/2	1	2.666	1.733	0.571	0.626	0.423	0.398
2/3	1	2	1.5	0.5	0.5	0.333	0.271
1	1.5	1.333	1.666	(*)	0.5	0.387	0.251

Table 2
Convergence factor of V-cycle (for $N_0 = 2$ and $J = 6$) and the corresponding bounds for $\nu = 2$; (*) the quantity exists, but does not correspond to the bound, since $\omega^{(1)} > 1$.

ω_{Jac}	$\omega^{(2)}$	$c_A^{(1)}$	$c_A^{(2)}$	$c_A^{(4)}$	$\frac{c_A^{(1)}}{c_A^{(1)}+4}$	$\frac{c_A^{(2)}}{c_A^{(2)}+2}$	$1-\frac{2-\omega^{(2)}}{K^{(2)}}$	$1-\delta^{(2)}$	$\rho(E_{MG}^{(J)})$
1/2	1	2.666	1.733	1.337	0.4	0.4	0.423	0.252	0.187
2/3	1	2	1.5	1.25	0.333	0.333	0.333	0.2	0.121
1	1	1.333	1.666	1.233	(*)	0.25	0.4	0.189	0.091

Note that for $s_l \to 0$ the third term is larger that the maximum over s_l of the first and the second; hence

$$\rho(0, s_l) \leqslant \sup_{s_l \in (0,1)} \left(\frac{s_l}{1-(1-\frac{3\omega_{\text{Jac}}}{2} s_l)^\nu} + \frac{1-s_l}{1-(1-\frac{3\omega_{\text{Jac}}}{2}(1-s_l))^\nu} \right). \tag{5.7}$$

The right-hand side of (5.7) is in fact independent of s_l for $\nu = 1$, and, for $\nu = 2$ and $\nu = 4$, one may check, using elementary function analysis (see Appendix B), that the supremum is reached for $s_l \to 0, 1$. Hence

$$c_A^{(\nu)} \leqslant \frac{2}{3\nu\omega_{\text{Jac}}} + \frac{1}{1-(1-\frac{3\omega_{\text{Jac}}}{2})^\nu}, \quad \nu = 1, 2, 4. \tag{5.8}$$

Using the relation (5.8) as an equality, we can evaluate the different bounds. This is done in Tables 1 and 2 for different number ν of smoothing steps, where we also compare the bounds with the actual convergence factor. One sees that McCormick's bound is indeed the best one and, further, that it gives in the considered cases a satisfactory sharp prediction of actual multigrid convergence.

Appendix A

We first show that Theorem 5.1 in [20] particularized to the matrix case (that is, applied to the case of matrix operators in $\mathbb{R}^n$ with $a(v, w) = (v, Aw) = v^T Aw$) yields the same bound as Theorem 3.1 (except for the additional refinement in the definition of $\|\Gamma\|$), provided that one has $\mathcal{W}_k = \mathcal{R}(\check{P}_k)$ and $\mathcal{V}_k = \mathcal{R}(\check{P}_k\check{G}_k - \check{P}_{k-1}\check{G}_{k-1})$, where $\check{P}_k$ and $\check{G}_k$ refer to the notation in Theorem 3.1, and $\mathcal{W}_k, \mathcal{V}_k$ to notation in [20].

Firstly, note that Theorem 5.1 provides a bound on the energy norm of product iteration matrices of the form (3.10), where

$$T_k = B_k^+ Q_k A, \tag{A.1}$$

B_k^+ being a matrix corresponding to an invertible operator onto $\mathcal{W}_k$, and Q_k being the orthogonal projector on the subspace $\mathcal{W}_k = \mathcal{R}(\check{P}_k)$; that is, $Q_k = \check{P}_k(\check{P}_k^T\check{P}_k)^{-1}\check{P}_k^T$. It then follows that the definition (A.1) matches (3.9) by setting $B_k^+ = \check{P}_k M_k^{-1}\check{P}_k^T$. Observe also that, $\forall w_k \in \mathcal{W}_k$,

$$z_k = B_k^+ w_k \quad \Leftrightarrow \quad w_k = \check{P}_k\big(\check{P}_k^T\check{P}_k\big)^{-1} M_k \big(\check{P}_k^T\check{P}_k\big)^{-1}\check{P}_k^T z_k.$$

Hence

$$B_k = \check{P}_k\big(\check{P}_k^T\check{P}_k\big)^{-1} M_k \big(\check{P}_k^T\check{P}_k\big)^{-1}\check{P}_k^T \tag{A.2}$$

is the proper inverse of B_k^+ onto $\mathcal{W}_k$.

Next, the bound on $\|F_J\|_A^2$ in [20] is based on the decomposition of any vector $v \in \mathbb{R}^n$ as

$$v = \sum_{k=0}^{J} v_k,$$

where $v_k \in \mathcal{V}_k$. With $\mathcal{V}_k = \mathcal{R}(\check{P}_k\check{G}_k - \check{P}_{k-1}\check{G}_{k-1})$, it means

$$v_k = \check{P}_k(I - P_{k-1}G_{k-1})\check{G}_k v = (\check{P}_k\check{G}_k - \check{P}_{k-1}\check{G}_{k-1})v. \tag{A.3}$$

Then, the bound in [20] is

$$\|F_J\|_A^2 \leqslant 1 - \frac{2-\omega}{K_1(1+K_2)^2}, \tag{A.4}$$

where K_1 is such that

$$\sum_{k=0}^{J}(B_k v_k, v_k) \leqslant K_1 v^T A v \quad \forall v \in \mathbb{R}^n, \tag{A.5}$$

where ω satisfy

$$(Aw_k, w_k) \leqslant \omega(B_k w_k, w_k) \quad \forall w_k \in \mathcal{W}_k,\ k = 1, \ldots, J, \tag{A.6}$$

and where $K_2 = \|\widetilde{\Gamma}\|$, with $\widetilde{\Gamma} = (\widetilde{\gamma}_{kl})$ being the $(J+1)\times(J+1)$ matrix whose coefficients are such that

$$(Aw_k, v_l) \leqslant \widetilde{\gamma}_{kl}(B_k w_k, w_k)^{1/2}(B_l v_l, v_l)^{1/2} \quad \forall v_k \in \mathcal{V}_k,\ w_k \in \mathcal{W}_k \tag{A.7}$$

for $k \leqslant l$, and $\widetilde{\gamma}_{kl} = \widetilde{\gamma}_{lk}$ for $k > l$.

With (A.2) and (A.3), it is easy to recognize that $K^{(\nu)}$ in (3.4) is the best constant K_1 satisfying (A.5). On the other hand, "$\forall w_k \in \mathcal{W}_k$" means "for all $w_k = \check{P}_k w$ with $w \in \mathbb{R}^n$" and "$\forall v_k \in \mathcal{V}_k$" means "for all $v_k = \check{P}_k(I - P_{k-1}G_{k-1})\check{G}_k v$ with $v \in \mathbb{R}^n$". Hence, for $k < l$, γ_{kl} in (3.6) is the best $\widetilde{\gamma}_{kl}$ satisfying (A.7). Further, using the same arguments, we see that $\omega^{(\nu)}$ is the best choice for ω. Therefore, the equivalence between the bound (A.4) in [20] and (3.3) is proved, except for the additional refinement showing that the lower triangular part of Γ can be set to zero.

We next show that with any admissible choice of $\mathcal{V}_k$, one may associate valid G_k, $k = 0, \ldots, J$, such that $\mathcal{V}_k = \mathcal{R}(\check{P}_k\check{G}_k - \check{P}_{k-1}\check{G}_{k-1})$ (setting $P_{-1} = G_{-1} = O_{n_0\times n_0}$). In other words, any bound from Theorem 5.1 in [20] obtained using a particular decomposition can also be obtained via (3.3) (up to some additional refinement in the definition of $\|\Gamma\|$) using a particular set of matrices G_k.

We begin the proof letting

$$\mathcal{X}_k = \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_k.$$

Observe that the proposition holds if, given $\mathcal{X}_0 \subset \mathcal{X}_1 \subset \cdots \subset \mathcal{X}_J = \mathbb{R}^n$, one can find G_k, $k = 0, \ldots, J$, such that

$$\mathcal{R}(\check{P}_k\check{G}_k) = \mathcal{X}_k \tag{A.8}$$

and

$$\mathcal{R}(\check{P}_k\check{G}_k - \check{P}_{k-1}\check{G}_{k-1}) \cap \mathcal{R}(\check{P}_{k-1}\check{G}_{k-1}) = \{0\}.$$

The latter equality is checked if, for all $v, w \in \mathbb{R}^n$,

$$(\check{P}_k\check{G}_k - \check{P}_{k-1}\check{G}_{k-1})v = \check{P}_{k-1}\check{G}_{k-1}w \quad\Rightarrow\quad (\check{P}_k\check{G}_k - \check{P}_{k-1}\check{G}_{k-1})v = \check{P}_{k-1}\check{G}_{k-1}w = 0;$$

that is, since $\check{P}_k$ has full rank, if

$$\begin{aligned}&(I - P_{k-1}G_{k-1})(\check{G}_k v) = P_{k-1}G_{k-1}(\check{G}_k w)\\ &\quad\Rightarrow\quad (I - P_{k-1}G_{k-1})(\check{G}_k v) = P_{k-1}G_{k-1}(\check{G}_k w) = 0.\end{aligned} \tag{A.9}$$

This proposition is true when $P_{k-1}G_{k-1}$ is a projector (note that $P_{-1}G_{-1} = O_{n_0\times n_0}$ is a projector as well). The right equalities (A.9) follow then from the multiplication of (A.9) by $(I - P_{k-1}G_{k-1})$ and $P_{k-1}G_{k-1}$, respectively.

We now assume that $\check{G}_j$ has been constructed properly for $j = J, \ldots, k+1$ (which holds trivially for $j = J-1$), and show that one can construct G_k such that

$$\mathcal{R}(\check{P}_k G_k \check{G}_{k+1}) = \mathcal{X}_k \tag{A.10}$$

while satisfying the constraint

$$G_k P_k G_k = G_k, \tag{A.11}$$

yielding the required result by induction, since (A.11) implies $(P_kG_k)^2 = P_kG_k$.

Let $m_k = \dim(\mathcal{X}_k)$. Observe that $\mathcal{W}_0 \subset \cdots \subset \mathcal{W}_k$ implies $m_k \leqslant \dim(\mathcal{W}_k) = n_k$. Hence (A.10) holds if $\mathcal{G}_k = \mathcal{R}(\check{G}_k)$ is a prescribed m_k-dimensional subspace of $\mathbb{R}^{n_k}$ whose image by $\check{P}_k$ is $\mathcal{X}_k$. Let H_k be an $n_k \times m_k$ matrix whose columns form a basis of this subspace. We search for G_k of the form

$$G_k = H_k Z_k,$$

where Z_k is an $m_k \times n_{k+1}$ matrix of rank m_k. Then (A.10) holds if $Z_k\check{G}_{k+1}$ has rank m_k, which is ensured if $\mathcal{R}(\check{G}_{k+1})$ contains an m_k-dimensional subspace complementary to $\mathcal{N}(Z_k)$ (see [12, p. 199]). Note that $\dim(\mathcal{R}(\check{G}_{k+1})) = \dim(\mathcal{X}_{k+1}) \geqslant m_k$, hence there exists at least one m_k-dimensional subspace $\mathcal{G}_k$ of $\mathcal{R}(\check{G}_{k+1})$, and we shall enforce the null space of Z_k to be complementary to $\mathcal{G}_k$.

Consider now the constraint (A.11). With the given form of G_k, it is satisfied when

$$Z_k P_k H_k = I_{m_k};$$

that is, according to the terminology in [2], if Z_k is a $\{1,2\}$-inverse of $P_k H_k$. As shown in [2, p. 59], given any subspace $\mathcal{S}_k$ complementary to $\mathcal{T}_k = \mathcal{R}(P_k H_k)$ there exist such a $\{1,2\}$-inverse having $\mathcal{S}_k$ as a null space.

Hence the required result is proven if one can always find $\mathcal{S}_k$ complementary to both $\mathcal{G}_k$ and $\mathcal{T}_k$. This, in turn, is true since $\mathcal{G}_k$ and $\mathcal{T}_k$ are subspaces of the same dimension of a finite-dimensional space $\mathbb{R}^{n_k}$, see [9].

Appendix B

Lemma B.1. *Let P_k, $k = 0, \ldots, J-1$, be $n_{k+1} \times n_k$ matrices of rank n_k with $n = n_J > n_{J-1} > \cdots > n_0$. Let G_k, $k = 0, \ldots, J-1$, be $n_{k+1} \times n_k$ matrices such that*

$$G_k P_k = I_{n_k}.$$

Set $P_{-1} = G_{-1} = O_{n_0 \times n_0}$ and let, for $k = 0, \ldots, J$, $\check{P}_k$ be defined by (3.1), *$\check{G}_k$ be defined by* (3.2), *and $\check{Q}_k = (I - P_{k-1}G_{k-1})\check{G}_k$.*

There holds, for $0 \leqslant l, k \leqslant J$ with $k \neq l$,

$$\check{Q}_k \check{P}_k \check{Q}_k = \check{Q}_k \quad \text{and} \quad \check{Q}_l \check{P}_k \check{Q}_k = O_{n_l \times n}.$$

Proof. Note that $G_k P_k = I_{n_k}$ implies $\check{G}_k \check{P}_k = I_{n_k}$. The first statement follows then from

$$\begin{aligned}(I - P_{k-1}G_{k-1})\check{G}_k \check{P}_k (I - P_{k-1}G_{k-1}) &= (I - P_{k-1}G_{k-1})(I - P_{k-1}G_{k-1}) \\ &= I - P_{k-1}G_{k-1}.\end{aligned}$$

To prove the second statement, we consider two cases. If $l > k$,

$$\begin{aligned}(I - P_{l-1}G_{l-1})\check{G}_l \check{P}_k &= (I - P_{l-1}G_{l-1})G_l \cdots G_{J-1}P_{J-1} \cdots P_l P_{l-1} \cdots P_k \\ &= (I - P_{l-1}G_{l-1})P_{l-1} \cdots P_k \\ &= P_{l-1}(I - G_{l-1}P_{l-1})P_{l-2} \cdots P_k \\ &= O_{n_l \times n_k},\end{aligned}$$

whereas, if $l < k$,

$$\begin{aligned}\check{G}_l \check{P}_k (I - P_{k-1}G_{k-1}) &= G_l \cdots G_{k-1}G_k \cdots G_{J-1}P_{J-1} \cdots P_k (I - P_{k-1}G_{k-1}) \\ &= G_l \cdots G_{k-1}(I - P_{k-1}G_{k-1}) \\ &= G_l \cdots G_{k-2}(I - G_{k-1}P_{k-1})G_{k-1} \\ &= O_{n_l \times n_k}. \qquad \square\end{aligned}$$

Appendix C

In this appendix we outline for even values of ν the proof of the following identity

$$\sup_{s_l \in (0,1)} \left(\frac{s_l}{1 - (1 - \frac{3\omega_{\mathrm{Jac}}}{2} s_l)^\nu} + \frac{1 - s_l}{1 - (1 - \frac{3\omega_{\mathrm{Jac}}}{2}(1 - s_l))^\nu} \right) = \frac{2}{3\nu\omega_{\mathrm{Jac}}} + \frac{1}{1 - (1 - \frac{3\omega_{\mathrm{Jac}}}{2})^\nu},$$

with $\omega_{\mathrm{Jac}} \in (0, 4/3)$. More precisely, we prove that

$$f(s_l) = \frac{s_l}{1 - (1 - \frac{3\omega_{\mathrm{Jac}}}{2} s_l)^\nu}$$

is a convex function for $\omega_{\mathrm{Jac}} \in (0, 4/3)$, and hence so is $f(s_l) + f(1 - s_l)$, the prove being finished by the fact that any convex function takes it supremum at the boundary.

Now, note that

$$\tilde{f}(c) = \frac{3\omega_{\mathrm{Jac}}}{2} f\big(c(2/3)\omega_{\mathrm{Jac}}^{-1}\big) = \big(1 + c + \cdots + c^{\nu-1}\big)^{-1} = g(c)^{-1}$$

is convex for $c \in (-1,1)$ if and only if $f(s_l)$ is convex. However, $\tilde{f}(c)$ is convex if $\frac{d^2\tilde{f}}{dc^2} > 0$ for $c \in (-1,1)$, that is, if $\frac{d^2 g}{dc^2} \cdot g < 2 \cdot (\frac{dg}{dc})^2$. On the other hand, one can check that

$$\frac{d^2 g}{dc^2} \cdot g - 2\left(\frac{dg}{dc}\right)^2 = -\sum_{i=0}^{\nu/2-1} c^{2i-2}\big(c^2 + i(\nu-2i)(c+1)^2\big),$$

this last term being negative for $c \in (-1,1)$.

References

[1] A. Aricò, M. Donatelli, S. Serra-Capizzano, V-cycle optimal convergence for certain (multilevel) structured linear systems, SIAM Journal on Matrix Analysis and Applications 26 (2004) 186–214.
[2] A. Ben Israel, T.N.E. Greville, Generalized Inverses: Theory and Applications, J. Wiley and Sons, New York, 1974.
[3] D. Braess, The convergence rate of a multigrid method with Gauss–Seidel relaxation for the Poisson equation, in: W. Hackbusch, U. Trottenberg (Eds.), Multigrid Methods, in: Lectures Notes in Mathematics, vol. 960, Springer-Verlag, Berlin/Heidelberg/New York, 1982, pp. 368–386.
[4] D. Braess, W. Hackbusch, A new convergence proof for the multigrid method including the V-cycle, SIAM Journal on Numerical Analysis 20 (1983) 967–975.
[5] J.H. Bramble, J.E. Pasciak, J. Wang, J. Xu, Convergence estimates for multigrid algorithms without regularity assumptions, Mathematics of Computation 57 (1991) 23–45.
[6] M. Griebel, P. Oswald, On the abstract theory of additive and multiplicative Schwarz algorithms, Numerische Mathematik 70 (1995) 163–180.
[7] W. Hackbusch, Multi-grid convergence theory, in: W. Hackbusch, U. Trottenberg (Eds.), Multigrid Methods, in: Lectures Notes in Mathematics, vol. 960, Springer-Verlag, Berlin/Heidelberg/New York, 1982, pp. 177–219.
[8] W. Hackbusch, Multi-grid Methods and Applications, Springer, Berlin, 1985.
[9] M. Lauzon, S. Treil, Common complements of two subspaces of a Hilbert space, Journal of Functional Analysis 212 (2003) 500–512.
[10] J. Mandel, S.F. McCormick, J.W. Ruge, An algebraic theory for multigrid methods for variational problems, SIAM Journal on Numerical Analysis 25 (1988) 91–110.
[11] S.F. McCormick, Multigrid methods for variational problems: general theory for the V-cycle, SIAM Journal on Numerical Analysis 22 (1985) 634–643.
[12] C.D. Meyer, Matrix Analysis and Applied Linear Algebra, SIAM, Philadelphia, PA, 2000.
[13] P. Oswald, Multilevel Finite Element Approximation: Theory and Applications, Teubner Skripte zur Numerik, Teubner, Stuttgart, 1994.
[14] P. Oswald, Subspace correction methods and multigrid theory, in: [16], pp. 533–572, Appendix A.
[15] J.W. Ruge, K. Stüben, Algebraic multigrid (AMG), in: S.F. McCormick (Ed.), Multigrid Methods, in: Frontiers in Applied Mathematics, vol. 3, SIAM, Philadelphia, PA, 1987, pp. 73–130.
[16] U. Trottenberg, C.W. Oosterlee, A. Schüller, Multigrid, Academic Press, London, 2001.
[17] P.S. Vassilevski, Multilevel Block Factorization Preconditioners, Springer, New York, 2008.
[18] J. Xu, Iterative methods by space decomposition and subspace correction, SIAM Review 34 (1992) 581–613.
[19] J. Xu, L.T. Zikatanov, The method of alternating projections and the method of subspace corrections in Hilbert space, J. Amer. Math. Soc. 15 (2002) 573–597.
[20] H. Yserentant, Old and new convergence proofs for multigrid methods, Acta Numerica 2 (1993) 285–326.