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Short Note

# An efficient solver for the equations of resistive MHD with spatially-varying resistivity

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**Abstract**

We regularize the variable coefficient Helmholtz equations arising from implicit time discretizations for resistive MHD, in a way that leads to a symmetric positive-definite system uniformly in the time step. Standard centered-difference discretizations in space of the resulting PDE leads to a method that is second-order accurate, and that can be used with multigrid iteration to obtain efficient solvers.


*Keywords:* Resistive MHD; Helmholtz equation; Multigrid

The magnetic field equations for resistive MHD are given by

$$\frac{\partial \vec{B}}{\partial t} + \nabla \cdot (\vec{v}\vec{B} - \vec{B}\vec{v}) = -\nabla \times (\eta \nabla \times \vec{B}), \tag{1}$$

$$\nabla \cdot \vec{B} = 0, \tag{2}$$

where $\eta$ is the spatially-varying resistivity. In semi-implicit methods for (1) and (2), one uses an implicit discretization of the time evolution of the magnetic field as a sub-step due to the resistive terms. This leads to solving linear systems obtained from discretizing in space the following system of equations.

$$\begin{aligned} &\frac{1}{\sigma}\vec{B} + \widehat{L}\vec{B} = \vec{f}, \\ &\widehat{L}\vec{B} \equiv \nabla \times (\eta \nabla \times \vec{B}), \end{aligned} \tag{3}$$

where $\vec{f}$ satisfies $\nabla \cdot \vec{f} = 0$. Solutions to (3) satisfy the divergence-free condition $\nabla \cdot \vec{B} = 0$. Straightforward discretizations of (3) lead to linear systems that, in the limit $\sigma \to \infty$, are not amenable to the application of

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geometric multigrid. $\widehat{L}$ vanishes when applied to a gradient of a scalar potential, leading to high wavenumber modes in the discretized problem that are not damped by point relaxation. This does not affect the existence of solutions of the original system of PDEs, since the divergence-free condition on $\vec{f}$ is a necessary and sufficient condition for solvability of (3) for $\sigma \to \infty$.

For the case of constant $\eta$, we can use the identity

$$\nabla \times \nabla \times \vec{B} = -\Delta\vec{B} + \nabla(\nabla \cdot \vec{B}) \tag{4}$$

and the divergence-free condition to observe that solutions to (3) also satisfy

$$\begin{aligned} &\frac{1}{\sigma}\vec{B} + \widetilde{L}\vec{B} = \vec{f}, \\ &\widetilde{L}\vec{B} \equiv -\eta\nabla\vec{B} \end{aligned} \tag{5}$$

and conversely. In this case, we can use standard discretizations of the discrete Laplacian, $\Delta$; and geometric multigrid is an efficient iterative method for solving the resulting linear system for all $\sigma > 0$.

In this paper, we present a generalization of the formulation (5) to the case of spatially-varying resistivities, based on the observation that $\vec{B}$ is a solution to (3) if and only if $\vec{B}$ is a solution to

$$\begin{aligned} &\frac{1}{\sigma}\vec{B} + L\vec{B} = \vec{f}, \\ &L\vec{B} \equiv \nabla \times (\eta\nabla \times \vec{B}) - \nabla(\eta\nabla \cdot \vec{B}). \end{aligned} \tag{6}$$

The addition of the $\nabla(\eta\nabla \cdot \vec{B})$ term is in the spirit of a method originating with Godunov [3] of adding multiples of $\nabla \cdot \vec{B}$ to the ideal MHD equations to regularize the hyperbolic structure, leading to methods for which the magnetic field equations are still satisfied if the divergence-free condition is satisfied at some initial time [2,4,5]. In this case, we are adding terms to the resistive operator to regularize the parabolic structure of the equations.

As is the case for (5), the operator $L$ in (6) is a symmetric positive-definite operator. This is most easily seen by taking the inner product of $L(\vec{B})$ with $\vec{B}$ and integrating by parts over $\mathbb{R}^D$:

$$\int_{\mathbb{R}^D} (\nabla \times (\eta\nabla \times \vec{B})) \cdot \vec{B} - (\nabla(\eta\nabla \cdot \vec{B})) \cdot \vec{B}\,\mathrm{d}x = \int_{\mathbb{R}^D} \eta(|\nabla \times \vec{B}|^2 + |\nabla \cdot \vec{B}|^2)\,\mathrm{d}x. \tag{7}$$

In $\mathbb{R}^D$, this expression vanishes only if $\vec{B}$ vanishes identically. In more complicated domains, it is possible to obtain nonzero vector fields for which both the divergence and curl vanish, but such vector fields are extremely smooth and will probably not have any negative impact on the performance of multigrid. In any case, we do not consider such problems here.

Typically, solutions obtained from solving the equations using standard centered-difference spatial discretizations satisfy some discretized form of the divergence-free condition up to truncation error. However, this is also the case for a large class of methods for ideal MHD [1,5,6], and therefore the algorithm proposed here can be used to extend those methods to the resistive case, as has been done for constant resistivity.

To derive our discretization, we first observe that the operator $L$ in (6) can be expressed as the divergence of vector-valued fluxes:

$$L\vec{B} = \sum_{d=1}^{D} \frac{\partial(\eta\vec{F}^d)}{\partial x_d}, \tag{8}$$

$$\vec{F}^d = \nabla B_d - \frac{\partial\vec{B}}{\partial x_d} - (\nabla \cdot \vec{B})\boldsymbol{e}^d. \tag{9}$$

Here, $\boldsymbol{e}^d$ is the unit vector in the $d$ direction. We assume a cell-centered discretization for $\vec{B}$ : $\vec{B}^h_{\boldsymbol{i}} \approx \vec{B}(\boldsymbol{i}h)$, where $\boldsymbol{i} \in \mathbb{Z}^D$ and $h$ is the mesh spacing. Then, $L\vec{B}$ is discretized as follows:

$$(L^h\vec{B})_{\boldsymbol{i}} = \frac{1}{h}\sum_{d=1}^{D}\left(\eta_{\boldsymbol{i}+\frac{1}{2}\boldsymbol{e}^d}\vec{F}^d_{\boldsymbol{i}+\frac{1}{2}\boldsymbol{e}^d} - \eta_{\boldsymbol{i}-\frac{1}{2}\boldsymbol{e}^d}\vec{F}^d_{\boldsymbol{i}-\frac{1}{2}\boldsymbol{e}^d}\right), \tag{10}$$

where $\boldsymbol{e}^d$ is the unit vector in the $d$ direction, $\eta_{\boldsymbol{i}+\frac{1}{2}\boldsymbol{e}^d} = \eta\left(\left(\boldsymbol{i}+\frac{1}{2}\boldsymbol{e}^d\right)h\right)$ and $\vec{F}^d_{\boldsymbol{i}+\frac{1}{2}\boldsymbol{e}^d}$ is computed by replacing the derivatives at a face in (9) by centered-differences at the face:

$$\left.\frac{\partial \vec{B}}{\partial x_d}\right|_{\left(\boldsymbol{i}+\frac{1}{2}\boldsymbol{e}^d\right)h} \approx \frac{1}{h}(\vec{B}_{\boldsymbol{i}+\boldsymbol{e}^d} - \vec{B}_{\boldsymbol{i}}),$$

$$\left.\frac{\partial \vec{B}}{\partial x_{d'}}\right|_{\left(\boldsymbol{i}+\frac{1}{2}\boldsymbol{e}^d\right)h} \approx \frac{1}{4h}\left(\vec{B}_{\boldsymbol{i}+\boldsymbol{e}^{d'}} + \vec{B}_{\boldsymbol{i}+\boldsymbol{e}^{d'}+\boldsymbol{e}^d} - \vec{B}_{\boldsymbol{i}-\boldsymbol{e}^{d'}} - \vec{B}_{\boldsymbol{i}-\boldsymbol{e}^{d'}+\boldsymbol{e}^d}\right), \quad d' \neq d.$$

In the case where $\eta$ is a constant, the above discretization reduces to the standard $2D+1$-point centered-difference discretization of the Laplacian.

To demonstrate the method we first compute the convergence rates of the solution. Given an exact solution $\vec{B}^e$ to (3) the solution error $\varepsilon^h$ is defined as

$$\varepsilon^h_{\boldsymbol{i}} = \vec{B}^h_{\boldsymbol{i}} - \vec{B}^e(\boldsymbol{i}h),$$

where $\vec{B}^h$ is the solution to

$$\frac{1}{\sigma}\vec{B}^h + L^h(\vec{B}^h) = \vec{f}^h, \vec{f}^h_{\boldsymbol{i}} = f(\boldsymbol{i}h).$$

For these tests we define the exact solution to be

$$\vec{B}^e = ((\sin(2\pi y) + \sin(2\pi z)), (\sin(2\pi x) + \sin(2\pi z)), (\sin(2\pi y) + \sin(2\pi x))),$$

$$\vec{f} \equiv \frac{1}{\sigma}\vec{B}^e + \nabla \times (\eta \nabla \times \vec{B}^e)$$

in three dimensions. In two dimensions, $\vec{B}^e = (\sin(2\pi y), \sin(2\pi x))$. The coefficient $\eta = 1 + 0.1(\sin(2\pi x) + \sin(2\pi y) + \sin(2\pi z))$ in three dimensions and $\eta = 1 + 0.1(\sin(2\pi x) + \sin(2\pi y))$ in two dimensions and $f$ is defined so that it is divergence-free. The domain of computation is the unit square/cube, with periodic boundary conditions and equally spaced grid points. Solution error results for the case $\frac{1}{\sigma} = 1$ are given in Figs. 1 and 2. Solution error results for the case $\frac{1}{\sigma} = 0$ are given in Figs. 3 and 4. For both cases, we see robust second-order convergence in $L^1$, $L^2$ and $L^\infty$ norms.

For MHD applications, it is important for stability considerations that the numerical divergence of the magnetic field also converges to zero. We present convergence results for a second-order accurate approximation to $\nabla \cdot \vec{B}$ given an initial analytically divergence-free magnetic field

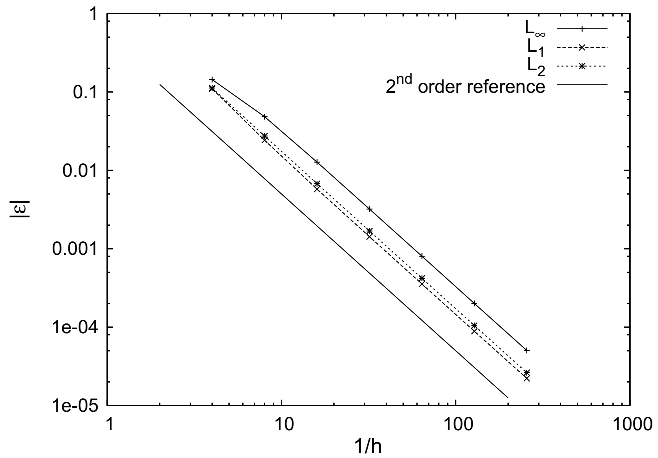


Fig. 1. Solution error at various resolutions in two dimensions, showing $O(h^2)$ convergence. $\frac{1}{\sigma} = 1$.

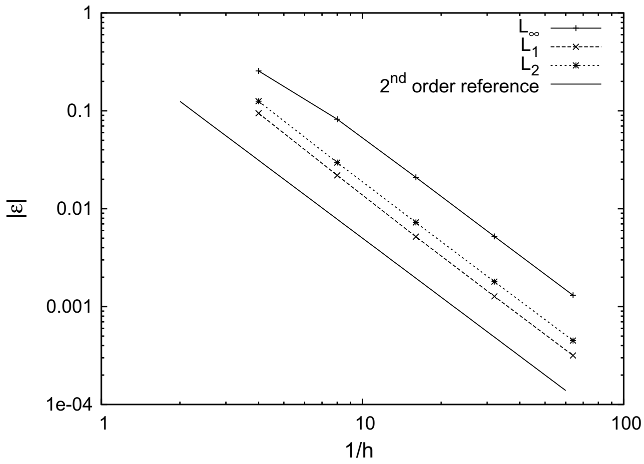


Fig. 2. Solution error at various resolutions in three dimensions, showing $O(h^2)$ convergence. $\frac{1}{\sigma} = 1$.

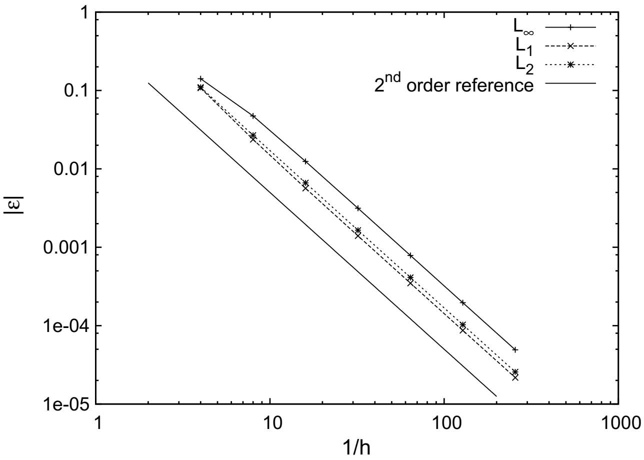


Fig. 3. Solution error at various resolutions in two dimensions, showing $O(h^2)$ convergence. $\frac{1}{\sigma} = 0$.

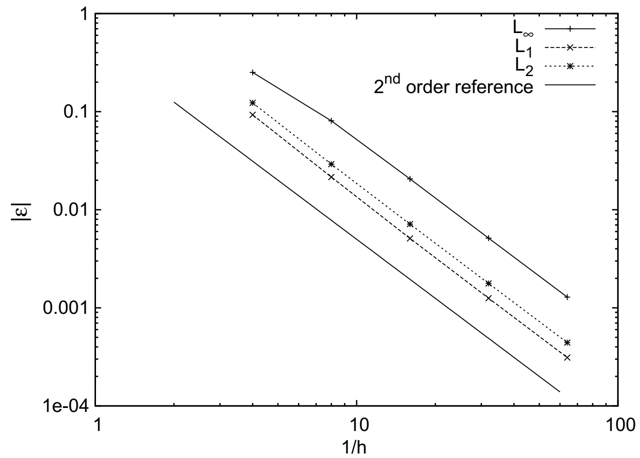


Fig. 4. Solution error at various resolutions in two dimensions, showing $O(h^2)$ convergence. $\frac{1}{\sigma} = 0$.

$$\nabla \cdot \vec{B} \approx \frac{1}{2h} \sum_{d=1}^{D} \left( \vec{B}_{i+\frac{1}{2}e^d} - \vec{B}_{i-\frac{1}{2}e^d} \right)$$

for the cases $\frac{1}{\sigma} = 1$ in Figs. 5 and 6 and results for $\frac{1}{\sigma} = 0$ in Figs. 7 and 8. We see that $\nabla \cdot \vec{B}$ converges to zero like $O(h^2)$ in $L^1$,$L^2$ and $L^\infty$ norms for both cases.

Figs. 9 and 10 compare multigrid convergence rates for the cases where the divergence of $\vec{B}$ is and is not included in the fluxes and $\frac{1}{\sigma} = 1$. Figs. 11 and 12 show multigrid convergence rates for the cases where the divergence of $\vec{B}$ is and is not included in the flux $F$ and $\frac{1}{\sigma} = 0$. Clearly, in both cases, the use of (6) instead of (3) makes the difference between multigrid converging or not.

We have demonstrated a technique for efficiently solving Helmholtz-like equations which appear in magnetohydrodynamics. Using the constraint that the magnetic field and the forcing terms are divergence-free we convert the resistive operator to a symmetric positive-definite operator by the addition of the term $-\nabla(\eta \nabla \cdot \vec{B})$. This approach, when discretized using standard finite volume methods for co-located field components, leads to efficient multigrid solvers, and the solution satisfies the divergence-free condition to truncation error. Multigrid iteration using the corresponding discretizations that do not include the additional divergence term fails to converge, or blows up after a few iterations.

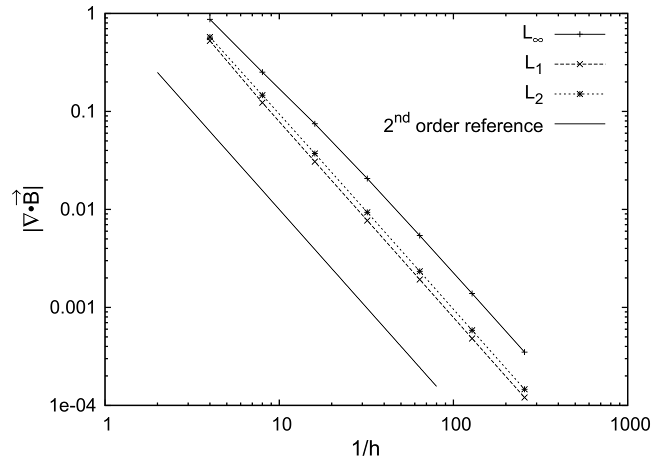


Fig. 5. $\nabla \cdot \vec{B}$ for solutions at various resolutions in two dimensions. $\nabla \cdot \vec{B}$ converges to zero at $O(h^2)$. $\frac{1}{\sigma} = 1$.

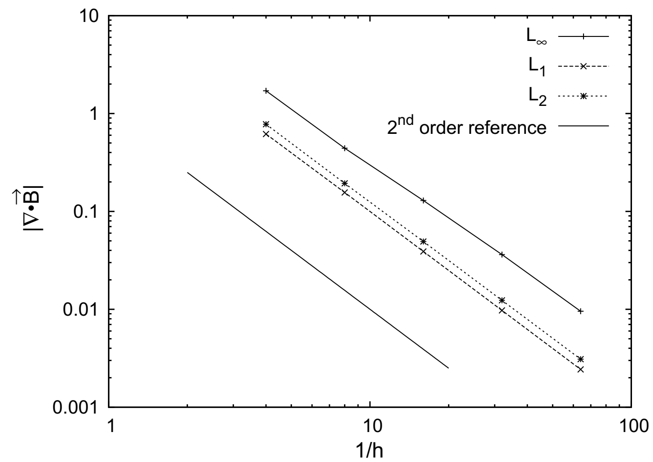


Fig. 6. $\nabla \cdot \vec{B}$ for solutions at various resolutions in three dimensions. $\nabla \cdot \vec{B}$ converges to zero at $O(h^2)$. $\frac{1}{\sigma} = 0$.

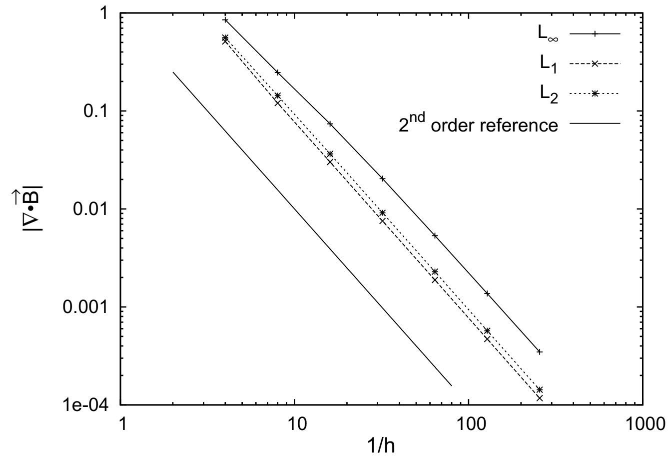


Fig. 7. $\nabla \cdot \vec{B}$ for solutions at various resolutions in two dimensions. $\nabla \cdot \vec{B}$ converges to zero at $O(h^2)$. $\frac{1}{\sigma} = 0$.

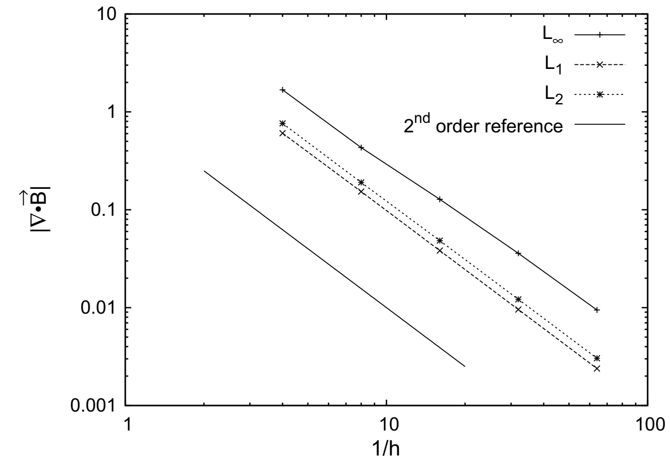


Fig. 8. $\nabla \cdot \vec{B}$ for solutions at various resolutions in three dimensions. $\nabla \cdot \vec{B}$ converges to zero at $O(h^2)$. $\frac{1}{\sigma} = 0$.

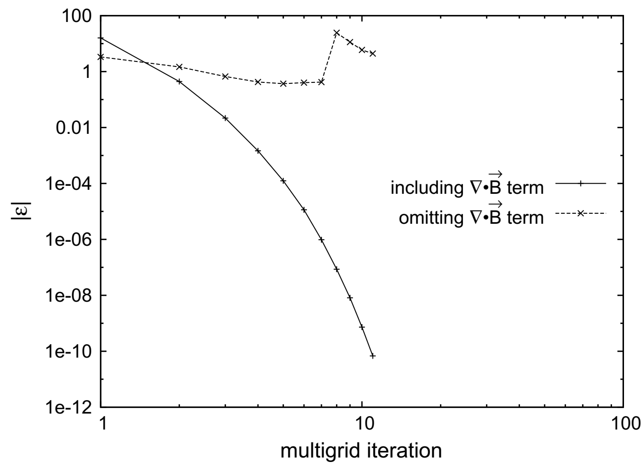


Fig. 9. Multigrid convergence in three dimensions with and without $\nabla \cdot \vec{B}$ term in the flux. $\frac{1}{\sigma} = 1$.

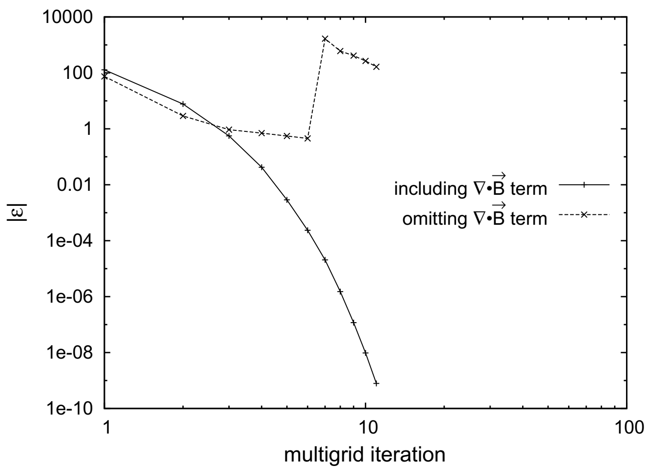


Fig. 10. Multigrid convergence in two dimensions with and without $\nabla \cdot \vec{B}$ term in the flux. $\frac{1}{\sigma} = 1$.

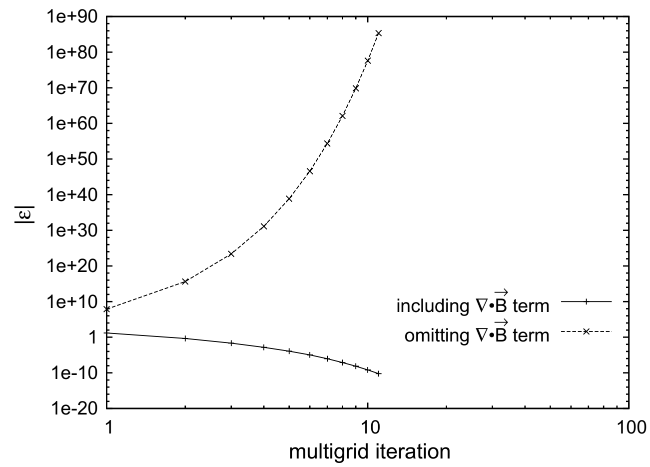


Fig. 11. Multigrid convergence in three dimensions with and without $\nabla \cdot \vec{B}$ term in the flux. $\frac{1}{\sigma} = 0$.

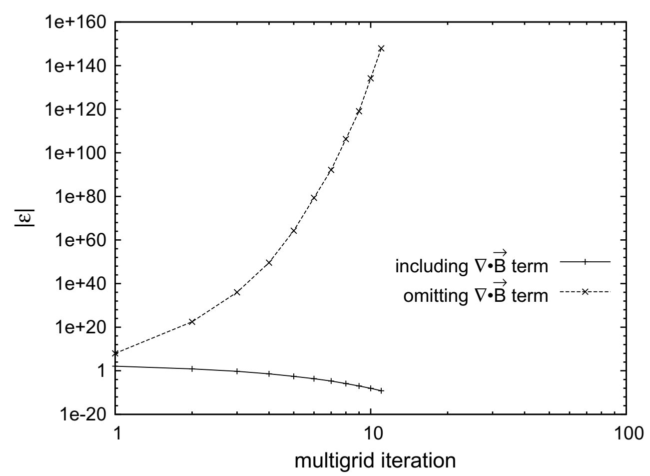


Fig. 12. Multigrid convergence in two dimensions with and without $\nabla \cdot \vec{B}$ term in the flux. $\frac{1}{\sigma} = 0$.

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