Cartesian Grid Embedded Boundary Methods for Solving Applied PDEs

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Overview

- Algorithms
  - Fundamental Concepts
  - Recent Developments

- Software
  - Design Overview
  - Grid Generation

- Results
  - AMR Hyperbolic Solver
  - Diffusion with a Moving Boundary
  - Wind over Mountains

- Future & Related Work

- Acknowledgments
Algorithms - Fundamental Concepts

• Consider PDEs written in conservation form:

\[ \nabla \cdot (\nabla \phi) = \nabla \cdot \vec{F} = \rho \]

\[ \frac{\partial U}{\partial t} + \nabla \cdot \vec{F}(U) = 0 \]

• Primary dependent variables approximate values at centers of Cartesian cells

• Divergence theorem over each control volume leads to “finite volume” approximation for \( \nabla \cdot \vec{F} \) (fluxes are at centroids):

\[ \nabla \cdot \vec{F} \approx \frac{1}{\kappa h^d} \int \nabla \cdot \vec{F} \, dx = \frac{1}{\kappa h} \sum \alpha_s \vec{F}_s \cdot \vec{n}_s + \alpha_B \vec{F} \cdot \vec{n}_B \equiv D \cdot \vec{F}^h \]
Algorithms - Fundamental Concepts (cont.)

- Away from the boundaries, method reduces to standard conservative finite difference method.
- \(D \cdot \vec{F}^h\) is consistent and accurate enough on cut cells to obtain 2nd order solution accuracy.
- \(D \cdot \vec{F}^h\) seems to be singular from both the standpoint of stability and accuracy - the elliptic and hyperbolic cases are handled differently.

\[
\nabla \cdot \vec{F} \approx \frac{1}{\kappa h^d} \int \nabla \cdot \vec{F} \, dx = \frac{1}{\kappa h} \sum \alpha_s \vec{F}_s \cdot \hat{n}_s + \alpha_B \vec{F} \cdot \hat{n}_B \equiv D \cdot \vec{F}^h
\]
Algorithms - Recent Developments

• Hyperbolic PDEs:
  – Consistent, conservative discretization with small control volume stability using weighted flux computation and redistribution

• Elliptic PDEs:
  – Consistent, stable flux calculation in 2D using linear interpolation and in 3D using bilinear interpolation

• Parabolic PDEs:
  – $2^{nd}$ order accurate, $L_0$-stable implicit Runge-Kutta (Twizell, Gumel, and Arigu - 1996)
  – Moving boundaries represented as a sequence of equivalent fixed boundary problems (McCorquodale, Colella, Johansen - 2001)
Software - Design Overview

We generalize rectangular array abstractions to represent more general graphs that map into the rectangular lattice $\mathbb{Z}^D$. The nodes of the graph are the control volumes, while the arcs of the graph are the faces across which fluxes are defined.

<table>
<thead>
<tr>
<th><strong>Index space</strong> = $\mathbb{Z}^D$</th>
<th><strong>EB Chombo</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Index (point) ∈ $\mathbb{Z}^D$</strong></td>
<td><strong>EBIndexSpace</strong></td>
</tr>
<tr>
<td><strong>Rectangular set of indices</strong></td>
<td><strong>VolIndex, FaceIndex</strong></td>
</tr>
<tr>
<td><strong>Rectangular array</strong></td>
<td><strong>EBISBox</strong></td>
</tr>
<tr>
<td></td>
<td>BaseEBCellFAB, BaseEBFaceFAB</td>
</tr>
</tbody>
</table>
Software - Grid Generation

- Cut cell generation via local function values:

- Cut cell generation via CAD description and Cart3D:
Results - AMR Hyperbolic Solver
Dan Graves

**Goal:** EB AMR Hyperbolic Solver using unsplit Godunov scheme

**Status:**
- 2D, $2^{nd}$ order accurate, x-y/r-z solver
- 3D, $2^{nd}$ order accurate solver underway

**Issues:**
- $2^{nd}$ order accurate, robust discretization of irregular control volume
- Performance - time and space
- Parallel load balancing
Results - Diffusion with a Moving Boundary
Peter Schwartz

Goal: Moving boundary with diffusion on interior and boundary with chemical coupling between boundary and interior

Status:
- 2nd order accurate interior diffusion with a stationary boundary
- 2nd order accurate interior diffusion with a moving boundary underway
- 1st order accurate coupling of chemistry using operator splitting

Issues:
- Geometry approximation consistent with the divergence theorem
- Stability of operator stencil in 3D
- Interactions between the interior and the surface
- Operator splitting to include chemistry and moving boundaries
Goal: 3D flow over mountains for climate prediction

Status:
- 2D anisotropic, non-hydrostatic, allspeed formulation (Colella and Pao - 1999) with gravity
- Testing and validation underway

Issues:
- Anisotropic multigrid with line relaxation
- Higher order intersection points between geometry and cell boundaries for scalar advection
- $2^{nd}$ order accurate, robust discretization of irregular control volume
Future & Related Work

• Future:
  – BIOMEMS
  – Environmental Simulations
  – Magnetohydrodynamics (MHD)

• Related:
  – Multifluid AMR
  – Particle AMR
Acknowledgments

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