A Strong Hot Spot Theorem

David H. Bailey and Daniel J. Rudolph

2 May 2003

In [3] Bailey and Richard Crandall established normality base b for the class of constants

$$\alpha_{b,c} = \sum_{k=1}^{\infty} \frac{1}{c^k b^{c^k}}, \tag{1}$$

where the integer $b > 1$ and c is odd and co-prime to b, as well as some generalizations of this class. The proof given in [3] is rather difficult and relies on several not-well-known results, including one by Korobov on the properties of certain pseudo-random sequences. Recently it has been shown that normality can be established much more easily, as a consequence of what may be called the "hot spot" theorem [1]. Here we state and prove a strong form of the "hot spot" theorem. A weaker result is given in [5, pg. 77], and is proven by an ergodic theory argument in [2].

In the following, $\{\cdot\}$ denotes fractional part as before, and $\#[\cdot]$ denotes count. μ and ν denote probability measures on U (the unit interval mod 1). $A - B$ denotes the set of $x \in A$ and $x \notin B$, and $A\Delta B = (A - B) \cup (B - A)$. The notation a.e. $x[\mu]$ means for all $x \in U$ except for a set Q with $\mu(Q) = 0$. A *Vitali covering* of a measurable set $A \subset U$ is a collection of open intervals with the property that every $x \in A$ is contained in infinitely many, arbitrarily small intervals in the collection. The measure ν is *absolutely continuous* with respect to μ if $\nu(A) = 0$ whenever $\mu(A) = 0$. The map $T : U \to U$ is said to be *measure-preserving* with respect to μ if $\mu(T^{-1}A) = \mu(A)$ for every μ-measurable set A, and *ergodic* with respect to μ if $T^{-1}A = A$ implies $\mu(A) = 0$ or 1.

Given a real constant α in $[0, 1)$, we define here a *base-b hot spot* to be some $x \in [0, 1)$ with the property that

$$\liminf_{h\to 0} \liminf_{n\to\infty} \frac{\#_{0\le j<n}[\{b^j\alpha\} \in (x - h, x + h)]}{2hn} = \infty. \tag{2}$$

Another way to state this condition is this: x is a base-b hot spot if given any $M > 0$, there is some $\delta_M > 0$ such that for all $h < \delta_M$ there is some $N_h > 0$ such that for all $n > N_h$, the condition $\#_{0\le j<n}[\{b^j\alpha\} \in (x - h, x + h)] > 2hnM$ holds.

What we shall establish below is that α is b-normal if and only if it has no base-b hot spots. We first present a few preliminary results.

Lemma 1 Vitali covering lemma. *If a μ-measurable set $A \subset U$ has a Vitali covering, then given any $\epsilon > 0$, there is some finite disjoint subcollection A' with the property that $\mu(A\Delta A') < \epsilon$.*

This result is proven in [6].

Lemma 2 Birkoff ergodic theorem. *Let $f(t)$ be an integrable function on $[0,1)$, and let T be an ergodic transformation for μ. Then*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(T^k x) \;=\; \int f\,d\mu \quad \textit{for a.e. } x[\mu], \tag{3}$$

This result is proved in [4, pg. 13, 20-29].

Lemma 3 Equivalence of absolutely continuous measures. *Suppose that T is measure-preserving and ergodic with respect to both μ and ν, and further that ν is absolutely continuous with respect to μ. Then $\mu = \nu$.*

Proof. Applying the ergodic theorem to $f(t) = I_A(t)$ (the indicator function of A),

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(T^k x) \;=\; \int f(t)\,d\mu(t) \;= \mu(A) \quad \text{for a.e. } x[\mu]. \tag{4}$$

Since ν is absolutely continuous with respect to μ, the above holds a.e. $x[\nu]$ as well. Now since T preserves the measure ν, we can write, for $n > 0$,

$$\begin{aligned} \nu(A) \;&=\; \int f(t)\,d\nu(t) \;=\; \frac{1}{n}\sum_{i=0}^{n-1}\int f(T^i x)\,d\nu(x) \\ &=\; \int \frac{1}{n}\sum_{i=0}^{n-1} f(T^i x)\,d\nu(x) \;\to\; \int \mu(A)d\nu \;=\; \mu(A), \end{aligned} \tag{5}$$

by the dominated convergence theorem. QED

Lemma 4 Absolute continuity of measures with finite derivatives. *Suppose ν is a measure on U with the property that for a.e. $x[\nu]$,*

$$\liminf_{h\to 0} \frac{\nu(x-h, x+h)}{2h} \;<\; \infty \tag{6}$$

Then ν is absolutely continuous with Lebesgue measure.

Proof. Here μ denotes Lebesgue measure on U, and ν denotes any measure as defined in the hypothesis. Let A be any set with $\mu(A) = 0$, and let $\epsilon > 0$ be given. Then there exists a set Q with $\nu(Q) < \epsilon$ and $M \geq 1$ such that the LHS of (6), as a function of x, is bounded by M except on Q. Further, there exists some open set $A' \supset A$ with $\mu(A') < \epsilon/M$. Then for every $x \in (A' - Q)$ there exists an infinite sequence h_k, strictly decreasing to zero, such that $(x - h_1, x + h_1) \subset A'$ and $\nu(x - h_k, x + h_k)/(2h_k) \leq M + \epsilon$ for $k \geq 1$. For $x \in (A' \cap Q)$, define $h_k = 2^{-m-k}$, where m is large enough that $(x - h_1, x + h_1) \subset A'$. Note that in either case all of these intervals are contained within A'. The collection of

these intervals is a Vitali covering of the set A', so by the Vitali covering lemma there is a finite disjoint subcollection $A'' \subset A'$ with $\nu(A' - A'') < \epsilon$. We can then write

$$\begin{aligned} \nu(A) &\leq \nu(A') = \nu(A'') + \nu(A' - A'') \\ &= \nu(A'' - Q) + \nu(A'' \cap Q) + \nu(A' - A'') \\ &\leq (M+\epsilon)\mu(A'' - Q) + 2\epsilon \leq (M+\epsilon)\mu(A') + 2\epsilon \\ &\leq (M+\epsilon)\epsilon/M + 2\epsilon < 4\epsilon, \end{aligned} \tag{7}$$

which implies that $\nu(A) = 0$. QED

In the following, μ will denote Lebesgue measure on U, and, given a real constant $\alpha \in U$ and an integer $b \geq 2$, ν will denote the measure defined on an interval (c, d) as

$$\nu(c, d) = \liminf_{n\to\infty} \frac{\#_{0\leq j<n}[\{b^j\alpha\} \in (c, d)]}{n} \tag{8}$$

Lemma 5 Ergodicity of the digit-shift transformation. *The digit-shift transformation $T(x) = \{bx\}$ is measure-preserving and ergodic with respect to both μ and ν.*

Proof. T clearly preserves Lebesgue measure. Assume for convenience that $b = 2$, and suppose that $A = T^{-1}(A)$. Then note that $x \in A$ if and only if $\{x + 1/2\} \in A$. Thus if $D = (0, 1/2)$, then $\mu(A \cap D) = \mu(A)/2 = \mu(A)\mu(D)$. A similar equality follows for any binary rational interval $(j2^m, k2^m)$, and thus for any finite disjoint union of such intervals. This collection of binary rational intervals is a Vitali covering of A. Thus given $\epsilon > 0$, there is some finite disjoint union E with $\mu(A\Delta E) < \epsilon$ and $\mu(A \cap E) = \mu(A)\mu(E)$. We can then write

$$\begin{aligned} |\mu(A) - \mu^2(A)| &< |\mu(A) - \mu(A)\mu(E)| + \epsilon = |\mu(A) - \mu(A \cap E)| + \epsilon \\ &= |\mu(A) - (\mu(A) - \mu(A - E))| + \epsilon \leq 2\epsilon \end{aligned} \tag{9}$$

Thus $\mu(A) = \mu^2(A)$, so that $\mu(A) = 0$ or 1 as required. A similar argument applies to the measure ν as defined above. In the parlance of ergodic theory, T is "mixing" with respect to both μ and ν, which condition is well-known to imply ergodicity [4, pg. 12]. QED

Theorem 1 Hot spot theorem. *The real constant α is b-normal if and only if it has no base-b hot spots.*

Proof. If α has no base-b hot spots, then it follows immediately from Lemmas 3, 4, and 5, that for any interval $(c, d) \subset U$,

$$\liminf_{n\to\infty} \frac{\#_{0\leq j<n}[\{b^j\alpha\} \in (c, d)]}{n} = \mu(c, d) = d - c \tag{10}$$

This result also applies to $(0,c) \cup (d,1)$, which except for c,d and the point 0 is the complement of (c,d). We can then write

$$\begin{aligned} \limsup_{n\to\infty} \frac{\#_{0\le j<n}[\{b^j\alpha\} \in (c,d)]}{n} &= 1 - \liminf_{n\to\infty} \frac{\#_{0\le j<n}[\{b^j\alpha\} \in (0,c)\cup(d,1)]}{n} \\ &= 1-(c+(1-d)) \;=\; d-c \end{aligned} \tag{11}$$

Thus the lim inf and the lim sup are identical. Since this holds for any interval (c,d), it holds in particular for any interval whose endpoints are of the form j/b^m. Thus α is b-normal. QED

References

[1] David H. Bailey, "A Hot-Spot Proof of Normality for the Alpha Constants," manuscript, 2003, available at the URL
`http://www.nersc.gov/~dhbailey/dhbpapers/alpha-normal.pdf`

[2] David H. Bailey and Daniel J. Rudolph, "An Ergodic Proof that Rational Times Normal is Normal," manuscript, 2002, available at the URL
`http://www.nersc.gov/~dhbailey/dhbpapers/ratxnormal.pdf`

[3] David H. Bailey and Richard E. Crandall, "Random Generators and Normal Numbers," *Experimental Mathematics*, vol. 11 (2002), pg. 527–546.

[4] Patrick Billingsley, *Ergodic Theory and Information*, John Wiley, New York, 1965.

[5] L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, Wiley-Interscience, New York, 1974.

[6] Halsey L. Royden, *Measure Theory*, Addison-Wesley, 1968.